

Drawing conclusions from ODEs

In our study of partial differential equations, we'll have occasion to need to know some fairly specific qualitative information about the solutions of certain second-order linear ordinary differential equations – in particular we'll need to know whether and how often the solutions become zero. It will be necessary to do this even in the absence of a specific formula for the solutions — very often, one just gives the solutions names (such as Bessel functions, Legendre functions, etc.) and then tries to make deductions about the functions.

It might seem remarkable at first, but it's possible to deduce all sorts of information about the solutions of an ODE without knowing precisely what they are. To illustrate this, we'll engage in a bit of a “thought experiment”. We're going to study the solutions of the familiar equation

$$y'' + y = 0$$

but we're going to “forget” that we know the solutions are linear combinations of $\sin x$ and $\cos x$. We do this to demonstrate just how much you can learn from the differential equation alone.

So here goes.

Because the equation $y'' + y = 0$ is a homogeneous second-order linear equation, we know to expect that every solution can be written as a linear combination of two *linearly independent* solutions. For our purposes, it's enough to say that two functions are linearly independent if neither can be expressed as a constant multiple of the other.

Our first task, then, is to come up with two linearly independent solutions — probably the most reasonable way to do this is to let one of them be the one having initial values $y(0) = 1$ and $y'(0) = 0$, and the other one have initial values $y(0) = 0$ and $y'(0) = 1$. This will guarantee that they're linearly independent.

Because we know full well what the solutions are going to turn out to be (but except for picking names we won't use any of this knowledge), we'll let $C(x)$ be the solution with $C(0) = 1$ and $C'(0) = 0$, and we'll let $S(x)$ satisfy $S(0) = 0$ and

$$S'(0) = 1.$$

What can we conclude about these functions? Let's think about $S(x)$ first – since $S(0) = 0$ and $S'(0) = 1$, we know that $S(x)$ is positive and increasing for small values of x . And because $S'' = -S$ from the differential equation, the graph of S is concave down, so the first derivative of S is becoming smaller and smaller. In fact, as long as $S' > 0$, S will get bigger and so S'' will get more negative and S' will decrease at an increasing rate. It is inevitable then that there will be a value of x , let's call it x_0 , where $S'(x_0) = 0$.

Now we'll get “clever” — since $S(x)$ is a solution of $S''(x) + S(x) = 0$, then so is $T(x) = S(2x_0 - x)$ (check it: $T'(x) = -S'(2x_0 - x)$ and $T''(x) = S''(2x_0 - x) = -S(2x_0 - x) = -T(x)$. using the chain rule and the fact that S satisfies the differential equation). But then both S and T satisfy the initial value problem

$$y'' + y = 0, \quad y(x_0) = S(x_0), \quad y'(x_0) = 0.$$

so $S(x)$ and $T(x) = S(2x_0 - x)$ are the same on the interval $x_0 < x < 2x_0$. This shows that the graph of $S(x)$ is symmetric around the vertical line $x = x_0$, and so $S(2x_0) = 0$ and $S'(2x_0) = -1$.

You can do the trick again with the function $-S(4x_0 - x)$ to show that between $2x_0$ and $4x_0$, the graph of S is the same as between 0 and $2x_0$, except reflected across the x -axis, and so $S(4x_0) = 0$ and $S'(4x_0) = 1$, so S is “back where it started” when $x = 4x_0$. You can use the trick over and over to show that S is periodic with period $4x_0$ (so apparently $x_0 = \pi/2$, but we won't use this fact).

What about C ? Well, before we think about C , let's observe that if we have any solution of $y'' + y = 0$, then then its derivative is also a solution (check it: if y is a solution, then plug y' into the equation: $(y')'' + (y') = y''' + y' = (y'' + y)' = 0$ because $y'' + y = 0$). But now we can see that the function $S'(x)$ satisfies the differential equation together with the initial conditions $S'(0) = 1$ and $S''(0) = -S(0) = 0$ – in other words S' satisfies the same initial conditions as C . Therefore we know that $C(x) = S'(x)$ for all x ; therefore we have that C is periodic with period $4x_0$ just like S is.

For our next trick, we'll use a useful quantity that comes up in the study of ODEs

called the *Wronskian* of two solutions of an ODE. If $y_1(x)$ and $y_2(x)$ are two solutions of a linear second-order equation, then their Wronskian $W(x)$ is the function

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

I bring up the Wronskian not so much to use it extensively as to point out that using the quantity $y_1y_2' - y_1'y_2$ doesn't come completely out of the blue.

Let's calculate the Wronskian of S and C , keeping in mind that $S' = C$:

$$W = S'C - C'S = C^2 - S''S = C^2 + S^2.$$

And while we're at it, note that

$$W' = 2CC' + 2SS' = 2CS'' + 2SC = 2C(-S) + 2SC = 0,$$

so W is a constant. What constant? Well, at zero, $S = 0$, $S' = 1$, $C = 1$, $C' = 0$, so $W(0) = 1$. So we've just proved that $S^2 + C^2 = 1$ for all x . You can use reasoning like this to prove pretty much all of the trig identities for sine and cosine in this fashion. Try your hand at the addition formulas:

$$S(a + b) = S(a)C(b) + S(b)C(a)$$

and

$$C(a + b) = C(a)C(b) - S(a)S(b).$$

From these you can get the double-angle formulas and then you've got it made.

About the only thing we haven't been able to do so far is figure out the value of x_0 — but we could do that by deriving the Taylor series for $C(x)$ from the differential equation and then estimating where it is zero to get an approximation of $\pi/2$, for example.

There will be some exercises so you can see how this kind of reasoning generalizes to a large class of linear second-order equations.