Math 425 **Dr. DeTurck**
Midterm 1 **February 8, 2007**

1. Solve $u_x + yu_y + u = 0$, $u(0, y) = y$. In what domain in the plane is your solution valid?

As usual, we construct the graph of the solution by propagating the initial data off the line in the xy-plane where the data are given, namely $x = 0$. To get the part of the solution emanating from the point $(0, b)$, we need to solve the system of ODEs:

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = y, \qquad \frac{du}{ds} + u = 0$$

together with initial data

$$x(0) = 0, \qquad y(0) = b, \qquad u(0) = b.$$

There is no "coupling" in this system, and each of the three ODEs can be solved by separation of variables to yield:

$$x(s; b) = s, \qquad y(x; b) = be^s, \qquad u(s; b) = be^{-s}.$$

So we have u as a function of b and s. We use the solutions for x and y to calculate b and s as functions of x and y:

$$s = x, \qquad b = \frac{y}{e^s} = \frac{y}{e^x}.$$

Now substitute this into the equation for $u(s; b)$ to get

$$u(x, y) = \frac{y}{e^x} e^{-x} = ye^{-2x}.$$

Since any point (x, y) can be expressed in terms of (b, s) via $x = s$, $y = be^s$, our solution is valid for all x and y.

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2. Let $u(x, t)$ be the temperature in a rod of length L that satisfies the partial differential equation:

$$u_t = ku_{xx}$$

for $(x, t) \in (0, L) \times (0, \infty)$, where k is a positive constant, together with the initial condition

$$u(x, 0) = \phi(x)$$

for $x \in [0, L]$, where ϕ satisfies $\phi(0) = \phi(L) = 0$ and $\phi(x) > 0$ for $x \in (0, L)$.

(a) If u also satisfies the Neumann boundary conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0,$$

show that the average temperature in the rod at time t, which is given by

$$A(t) = \frac{1}{L} \int_0^L u(x, t)\, dx$$

is a constant (independent of t).

(b) On the other hand, if u satisfies the Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

show that it must be the case the $u(x,t) \geq 0$ for all (x,t) satisfying $0 < x < L$ and $t > 0$.

(c) Still under the assumption that u satisfies the Dirichlet boundary conditions, show that $A(t)$ is a *non-increasing* function of t. (Hint for (a) and (c): Use an argument similar to an energy argument).

(a) To show that $A(t)$ is a constant, we calculate its derivative with respect to t, and use that u satisfies the PDE $u_t = ku_{xx}$:

$$\frac{dA}{dt} = \frac{1}{L}\int_0^L u_t(x,t)\,dx = \frac{k}{L}\int_0^T u_{xx}(x,t)\,dx = \frac{k}{L}(u_x(L,t) - u_x(0,t)).$$

But then the Neumann boundary conditions imply that both terms in the last difference are zero. So $A'(t) = 0$, hence A is constant.

(b) Because $u(0,t) = u(L,t) = 0$, and $u(x,0) = \phi(x) > 0$ for $x \in (0,L)$, it's clear that the minimum value of u on the standard "U"-shaped set in the maximum principle is zero. By the minimum principle (or the maximum principle applied to $-u$), the minimum of u for all (x,t) with $t > 0$ and $x \in (0,L)$ must be bigger than or equal to zero.

(c) From part (a), we already know that

$$\frac{dA}{dt} = \frac{k}{L}(u_x(L,t) - u_x,(0,t)).$$

And from part (b) we know that $u(x,t) \geq 0$ for $x \in (0,L)$ (in fact, if we use the *strong* version of the maximum principle, we know that $u(x,t) > 0$ for $x \in (0,L)$). So $u_x(L,t)$ cannot be positive (or else u would be increasing to zero as $x \to L$ from the left, contradicting $u \geq 0$) and $u_x(0,t)$ cannot be negative (for a similar reason). Therefore $dA/dt \leq 0$, and so A is non-increasing.

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3. (a) Solve the wave equation with friction: $u_{xx} = u_{tt} + 2u_t$ for $0 < x < \pi$ and $t > 0$ with the initial conditions $u(x,0) = \sin x$, $u_t(x,0) = 0$, and the boundary conditions $u(0) = u(\pi) = 0$. (Hint: Look for "separated solutions")

   (b) If

$$E(t) = \frac{1}{2}\int_0^\pi u_t^2 + u_x^2\,dx,$$

   show that

$$\lim_{t\to\infty} E(t) = 0.$$

   (Hint: To do this, you can calculate $E(t)$ explicitly).

(a) As usual, write $u(x,t) = X(x)T(t)$. Substituting into the equation and dividing by $X(x)T(t)$ gives us

$$\frac{X''(x)}{X(x)} = \frac{T''(t) + 2T'(t)}{T(t)} = \lambda$$

where λ is a constant. We'll short-circuit the process (see the solutions to the other set of practice problems) and notice that for $\lambda = -1$, we have $X'' = -X$, for which $X(x) = \sin x$ is a solution having $X(0) = X(\pi) = 0$.

Now we turn our attention to $T'' + 2T' = -T$, which has solutions $T = c_1 e^{-t} + c_2 t e^{-t}$. Our separated solution is thus

$$u(x,t) = \sin x(c_1 e^{-t} + c_2 t e^{-t})$$

and we have to choose c_1 and c_2 so that $u(x,0) = \sin x$ and $u_t(x,0) = 0$. The first condition certainly gives $c_1 = 1$. We calculate:

$$u_t(x,t) = \sin x(-e^{-t} + c_2 e^{-t} - c_2 t e^{-t})$$

which, when $t = 0$, gives $c_2 = 1$ as well. So the solution of the whole problem is

$$u(x,t) = \sin x(1+t)e^{-t}.$$

(b) Since $u_x = \cos x(1+t)e^{-t}$ and $u_t = -\sin x(te^{-t})$, it's clear that $u_t^2 + u_x^2$ has a factor of e^{-2t}. So, even without calculating the integral exactly, it's clear that $E(t) = e^{-2t}$ times a quadratic polynomial in t. So the limit of $E(t)$ as $t \to \infty$ is zero by l'Hospital's rule.

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4. Find as general a solution $u(x,y,z)$ as you can to the third-order equation

$$u_{xyz} = 0$$

Integrate first with respect to x to get u_{yz} is a constant plus a function depending only on y and z (the latter is the constant of integration with respect to x) – and continue in this way. Eventually end up with:

$$u(x,y,z) = F(x,y) + G(x,z) + H(y,z)$$

for three arbitrary functions F, G and H of two variables each.