1. Solve \( u_x + y u_y + u = 0, \ u(0, y) = y \). In what domain in the plane is your solution valid?

As usual, we construct the graph of the solution by propagating the initial data off the line in the \( xy \)-plane where the data are given, namely \( x = 0 \). To get the part of the solution emanating from the point \((0, b)\), we need to solve the system of ODEs:

\[
\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = y, \quad \frac{du}{ds} + u = 0
\]

together with initial data

\[
x(0) = 0, \quad y(0) = b, \quad u(0) = b.
\]

There is no “coupling” in this system, and each of the three ODEs can be solved by separation of variables to yield:

\[
x(s; b) = s, \quad y(x; b) = be^s, \quad u(s; b) = be^{-s}.
\]

So we have \( u \) as a function of \( b \) and \( s \). We use the solutions for \( x \) and \( y \) to calculate \( b \) and \( s \) as functions of \( x \) and \( y \):

\[
s = x, \quad b = \frac{y}{e^x} = \frac{y}{e^s}.
\]

Now substitute this into the equation for \( u(s; b) \) to get

\[
u(x, y) = \frac{y}{e^x} e^{-s} = ye^{-2x}.
\]

Since any point \((x, y)\) can be expressed in terms of \((b, s)\) via \( x = s \), \( y = be^s \), our solution is valid for all \( x \) and \( y \).

2. Let \( u(x, t) \) be the temperature in a rod of length \( L \) that satisfies the partial differential equation:

\[
u_t = ku_{xx}
\]

for \((x, t) \in (0, L) \times (0, \infty)\), where \( k \) is a positive constant, together with the initial condition

\[
u(x, 0) = \phi(x)
\]

for \( x \in [0, L] \), where \( \phi \) satisfies \( \phi(0) = \phi(L) = 0 \) and \( \phi(x) > 0 \) for \( x \in (0, L) \).

(a) If \( u \) also satisfies the Neumann boundary conditions

\[
u_x(0, t) = 0, \quad u_x(L, t) = 0,
\]

show that the average temperature in the rod at time \( t \), which is given by

\[
A(t) = \frac{1}{L} \int_0^L u(x, t) \, dx
\]

is a constant (independent of \( t \)).
(b) On the other hand, if \( u \) satisfies the Dirichlet boundary conditions
\[
u(0, t) = 0, \quad u(L, t) = 0,
\]
show that it must be the case the \( u(x, t) \geq 0 \) for all \((x, t)\) satisfying \( 0 < x < L \) and \( t > 0 \).

(c) Still under the assumption that \( u \) satisfies the Dirichlet boundary conditions, show that \( A(t) \) is a non-increasing function of \( t \). (Hint for (a) and (c): Use an argument similar to an energy argument).

(a) To show that \( A(t) \) is a constant, we calculate its derivative with respect to \( t \), and use that \( u \) satisfies the PDE \( u_t = ku_{xx} \):
\[
\frac{dA}{dt} = \frac{1}{L} \int_0^L u_t(x, t) \, dx = \frac{k}{L} \int_0^T u_{xx}(x, t) \, dx = \frac{k}{L} (u_x(L, t) - u_x(0, t)).
\]
But then the Neumann boundary conditions imply that both terms in the last difference are zero. So \( A'(t) = 0 \), hence \( A \) is constant.

(b) Because \( u(0, t) = u(L, t) = 0 \), and \( u(x, 0) = \phi(x) > 0 \) for \( x \in (0, L) \), it’s clear that the minimum value of \( u \) on the standard “U”-shaped set in the maximum principle is zero. By the minimum principle (or the maximum principle applied to \( -u \)), the minimum of \( u \) for all \((x, t)\) with \( t > 0 \) and \( x \in (0, L) \) must be bigger than or equal to zero.

(c) From part (a), we already know that
\[
\frac{dA}{dt} = \frac{k}{L} (u_x(L, t) - u_x(0, t)).
\]
And from part (b) we know that \( u(x, t) \geq 0 \) for \( x \in (0, L) \) (in fact, if we use the strong version of the maximum principle, we know that \( u(x, t) > 0 \) for \( x \in (0, L) \)). So \( u_x(L, t) \) cannot be positive (or else \( u \) would be increasing to zero as \( x \to L \) from the left, contradicting \( u \geq 0 \)) and \( u_x(0, t) \) cannot be negative (for a similar reason). Therefore \( dA/dt \leq 0 \), and so \( A \) is non-increasing.

3. (a) Solve the wave equation with friction: \( u_{xx} = u_{tt} + 2u_t \) for \( 0 < x < \pi \) and \( t > 0 \) with the initial conditions \( u(x, 0) = \sin x \), \( u_t(x, 0) = 0 \), and the boundary conditions \( u(0) = u(\pi) = 0 \). (Hint: Look for “separated solutions”)

(b) If
\[
E(t) = \frac{1}{2} \int_0^\pi u_t^2 + u_x^2 \, dx,
\]
show that
\[
\lim_{t \to \infty} E(t) = 0.
\]
(Hint: To do this, you can calculate \( E(t) \) explicitly).

(a) As usual, write \( u(x, t) = X(x)T(t) \). Substituting into the equation and dividing by \( X(x)T(t) \) gives us
\[
\frac{X''(x)}{X(x)} = \frac{T''(t) + 2T'(t)}{T(t)} = \lambda
\]
where \( \lambda \) is a constant. We’ll short-circuit the process (see the solutions to the other set of practice problems) and notice that for \( \lambda = -1 \), we have \( X'' = -X \), for which \( X(x) = \sin x \) is a solution having \( X(0) = X(\pi) = 0 \).
Now we turn our attention to $T'' + 2T' = -T$, which has solutions $T = c_1 e^{-t} + c_2 te^{-t}$. Our separated solution is thus

$$u(x,t) = \sin x (c_1 e^{-t} + c_2 te^{-t})$$

and we have to choose $c_1$ and $c_2$ so that $u(x,0) = \sin x$ and $u_t(x,0) = 0$. The first condition certainly gives $c_1 = 1$. We calculate:

$$u_t(x,t) = \sin x (-e^{-t} + c_2 e^{-t} - c_2 te^{-t})$$

which, when $t = 0$, gives $c_2 = 1$ as well. So the solution of the whole problem is

$$u(x,t) = \sin x (1 + t) e^{-t}.$$  

(b) Since $u_x = \cos x (1 + t) e^{-t}$ and $u_t = -\sin x (te^{-t})$, it’s clear that $u_t^2 + u_x^2$ has a factor of $e^{-2t}$. So, even without calculating the integral exactly, it’s clear that $E(t) = e^{-2t}$ times a quadratic polynomial in $t$. So the limit of $E(t)$ as $t \to \infty$ is zero by l’Hospital’s rule.

4. Find as general a solution $u(x,y,z)$ as you can to the third-order equation

$$u_{xyz} = 0$$

Integrate first with respect to $x$ to get $u_{yz}$ is a constant plus a function depending only on $y$ and $z$ (the latter is the constant of integration with respect to $x$) – and continue in this way. Eventually end up with:

$$u(x,y,z) = F(x,y) + G(x,z) + H(y,z)$$

for three arbitrary functions $F$, $G$ and $H$ of two variables each.