Math 425 **Dr. DeTurck**
Hints and Solutions to Practice Midterm 1 **February 2010**

1. Suppose f is a function of one variable that has a continuous second derivative. Show that for any constants a and b, the function
$$u(x,y) = f(ax+by)$$
is a solution of the PDE
$$u_{xx}u_{yy} - u_{xy}^2 = 0.$$

This is an exercise in using the chain rule. For instance, $u_x(x,y) = af'(ax+by)$, $u_{xx} = a^2 f''(ax+by)$, etc., so eventually:
$$u_{xx}u_{yy} - u_{xy}^2 = a^2b^2(f''(ax+by))^2 - (ab)^2(f''(ax+by))^2 = 0.$$

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2. Give an example that shows why solutions of the wave equation $u_{tt} = u_{xx}$ do *not* necessarily satisfy the maximum principle (i.e., give an example of an explicit solution of the equation for which the maximum principle does not hold).

For this, we need a solution to the wave equation for $x \in (0,L)$ and for $t \in (0,T)$ for which the maximum occurs in the interior of the rectangle. For instance, the function $u(x,t) = \sin x \sin t$ satisfies the wave equation, but the maximum of $u = 1$ occurs when $x = t = \pi/2$, in the interior of the rectangle $[0,\pi] \times [0,\pi]$ (where $u = 0$ identically on the boundary of the rectangle).

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3. Find the function $u(x,t)$ that satisfies
$$u_t = 2u_{xx}$$
for $(x,t) \in (0,3) \times (0,\infty)$, together with the initial condition
$$u(x,0) = \sin\frac{\pi x}{6} + 4\sin\frac{5\pi x}{6}$$
for $x \in [0,3]$, and the boundary conditions:
$$u(0,t) = 0 \qquad u_x(3,t) = 0$$
for all $t > 0$. (Hint: Look for "separated" solutions.)

A separated solution is of the form $u(x,t) = X(x)T(t)$, and we would need $X(0) = 0$ and $X'(3) = 0$ to satisfy the boundary conditions. For u of this form, the heat equation becomes:
$$X(x)T'(t) = 2X''(x)T(t).$$
Divide both sides by $2X(x)T(t)$ and get
$$\frac{T'(t)}{2T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side is a function of t alone, and the right side is a function of x alone, both sides must be constant, λ. Work on X first:

We have $X'' = \lambda X$, and either $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. If $\lambda > 0$, say $\lambda = k^2$, then $X(x) = c_1 e^{kx} + c_2 e^{-kx}$. The condition $X(0) = 0$ implies that $c_1 = -c_2$. Next, $X'(3) = 0$ means $kc_1 e^{3k} - kc_2 e^{-3k} = 0$, in other words $kc_1(e^{3k} + e^{-3k}) = 0$. But this implies $c_1 = 0$, so there's no non-zero solution of this form.

If $\lambda = 0$, then $X(x) = c_1 + c_2 x$, and then $F(0) = 0$ implies $c_1 = 0$ and $F'(3) = 0$ implies that $c_2 = 0$, so no non-zero solution here either.

Finally, if $\lambda < 0$, say $\lambda = -k^2$, then $X(x) = c_1 \sin(kx) + c_2 \cos(kx)$. $X(0) = 0$ implies $c_2 = 0$, and $X'(3) = 0$ means $kc_1 \cos(3k) = 0$, which is satisfied for $k = \pi/6, \pi/2, 5\pi/6, \ldots$. This is what we need.

If we use $k = \pi/6$ (so $\lambda = -\pi^2/36$), then we have $X(x) = \sin(\pi x/6)$ and $T(t)$ should satisfy $T' = -\pi^2 T/18$. So T is a constant times $e^{-\pi^2 t/18}$. Therefore, the separated solution

$$u_1(x,t) = e^{-\pi^2 t/18} \sin(\pi x/6)$$

satisfies the heat equation, the boundary conditions, and has initial data equal to the first term of the given initial data.

We can do the same thing with the second term and get that

$$u_2(x,t) = 4e^{-25\pi^2 t/18} \sin(5\pi/6)$$

works for that. And the sum $u(x,t) = u_1(x,t) + u_2(x,t)$ will be the solution of the whole problem

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4. Find the closed form (similar to d'Alembert's formula) of the solution $u(x,t)$ of the initial-boundary value problem for the semi-infinite string:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } x, t > 0$$

where $u(x,0) = f(x)$ for $x > 0$, and $u_t(x,0) = 0$ for $x > 0$, and $u(0,t) = \alpha(t)$ for $t \geq 0$, where f and α are C^2 functions and satisfy $f(0) = \alpha(0)$, $\alpha'(0) = 0$ and $\alpha''(0) = c^2 f''(0)$. Verify that the solution is C^2 for all $x, t > 0$.

We'll solve two separate problems here. First, we'll find $v(x,t)$ that satisfies everything except that $v(0,t) = 0$ instead of $\alpha(t)$. Then we'll find $w(x,t)$ that satisfies everything except that $v(x,0) = 0$ instead of $f(x)$. Then it'll be the case that $u(x,t) = v(x,t) + w(x,t)$ is the solution of the whole problem.

First, for $v(x,t)$, start with the d'Alembert form $v(x,t) = F(x+ct) + G(x-ct)$. We need to know values of $F(z)$ for $z > 0$, and of $G(z)$ for both positive and negative values of z. We have to reconcile this with $v(x,0) = f(x)$ for $x > 0$, $v_t(x,0) = 0$ and $v(0,t) = 0$ for $t > 0$. These conditions tell us:

$$F(x) + G(x) = f(x) \qquad \text{and} \qquad F'(x) - G'(x) = 0$$

for $x > 0$, and

$$G(-t) = G(t)$$

for $t > 0$. But from this it's clear that we should take $F(x) = G(x) = \frac{1}{2}f(x)$ for $x > 0$, and $G(s) = -\frac{1}{2}f(-s)$ if $x < 0$. This gives us:

$$v(x,t) = \begin{cases} \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) & \text{if} \quad x - ct > 0 \\ \\ \frac{1}{2}f(x+ct) - \frac{1}{2}f(ct-x) & \text{if} \quad x - ct < 0 \end{cases}$$

The interpretation of this is that the signal "bounces off" the fixed end of the string at $x = 0$ and is reflected back in "inverted" form.

Next, for $w(x,t)$, start with $v(x,t) = F(x+ct) + G(x-ct)$ as usual, where this time we need $w(x,0) = 0$ and $w_t(x,0) = 0$ for $x > 0$ and $w(0,t) = \alpha(t)$ for $t > 0$. These conditions tell us:

$$F(x) + G(x) = 0 \qquad \text{and} \qquad F'(x) - G'(x) = 0$$

for $x > 0$, so choose $F(x) = G(x) = 0$ for $x > 0$, and

$$G(-ct) = \alpha(t)$$

for $t > 0$, i.e., $G(s) = \alpha(-s/c)$ for $s < 0$. This gives us

$$w(x,t) = \begin{cases} 0 & \text{if} \quad x - ct > 0 \\ \\ \alpha(t - x/c) & \text{if} \quad x - ct < 0 \end{cases}$$

So altogether:

$$u(x,t) = \begin{cases} \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) & \text{if} \quad x - ct > 0 \\ \\ \frac{1}{2}f(x+ct) - \frac{1}{2}f(ct-x) + \alpha(t - x/c) & \text{if} \quad x - ct < 0 \end{cases}$$

That's it.