What am I going to do with all this symmetry??

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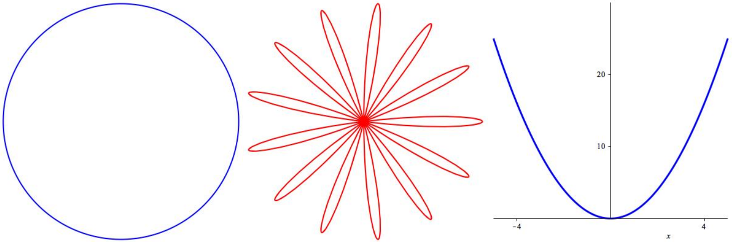
Symmetry

We're all familiar with the idea of symmetry — but can we define it?



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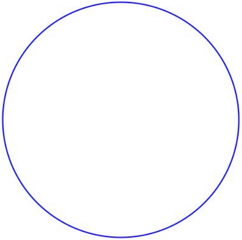
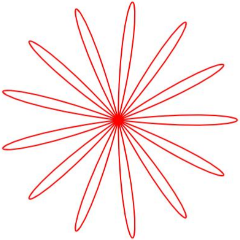
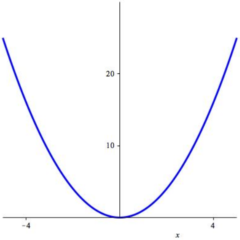


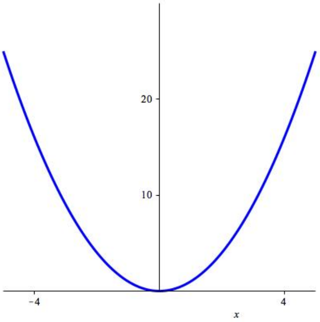




A symmetry occurs when there is a mapping (or transformation) of an object that leaves some property of the object unchanged. Symmetries can be discrete or continuous.

The parabola $y = x^2$

Let's look at the parabola $y = x^2$ again — what kind of symmetry does it have?



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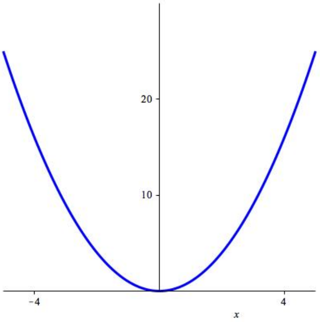
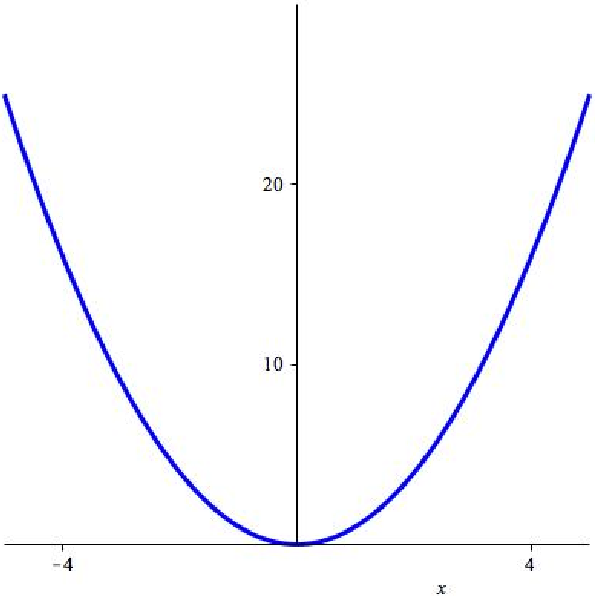


On the next few slides we'll draw several pictures of the parabola.

The parabola $y = x^2$



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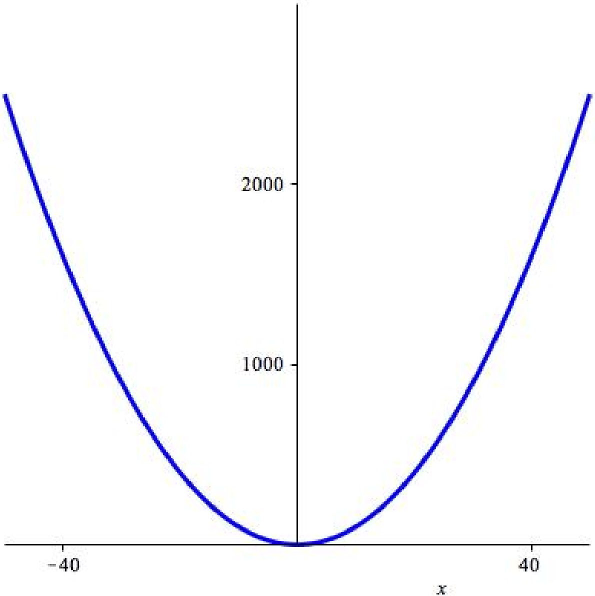

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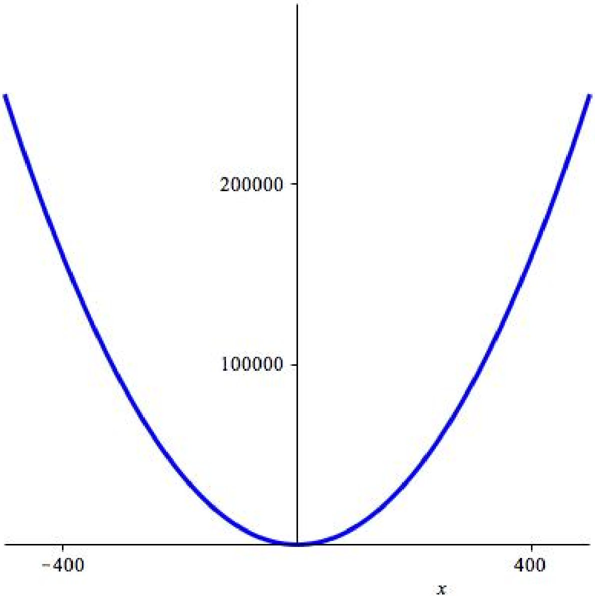

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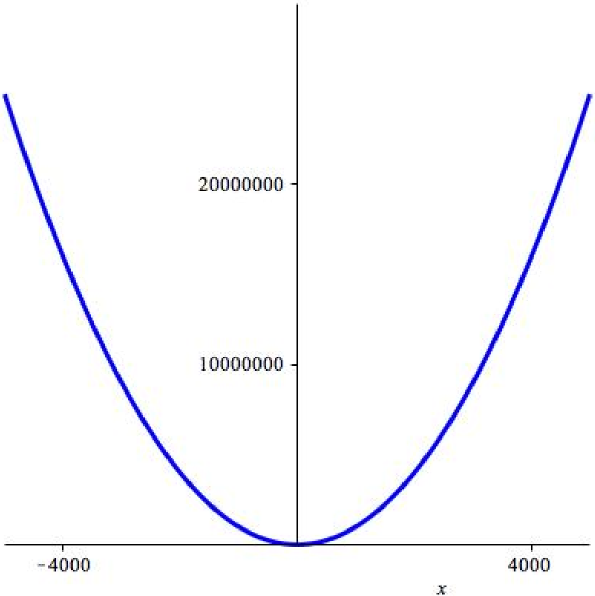

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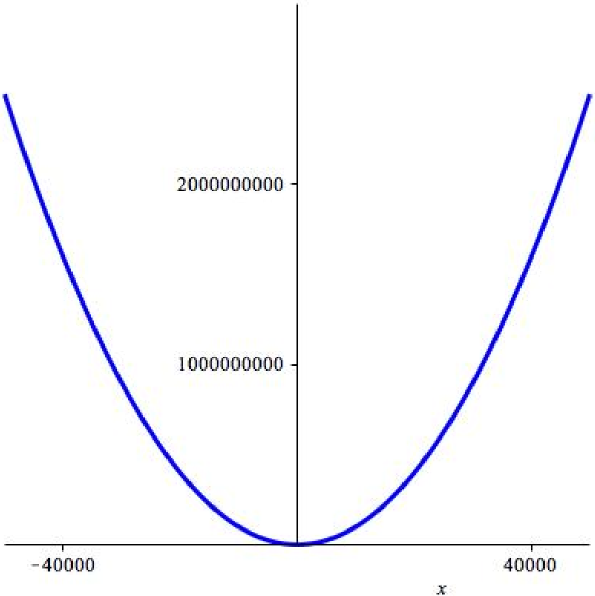

What changed?

What changed?

Only the scales on the axes!

In particular, the x values got multiplied by 10 and the y values by 100 each time.

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In particular, the x values got multiplied by 10 and the y values by 100 each time.

More generally, if we multiply the x values by k, we should multiply the y values by k^2 to preserve the picture. Since this works for all (real) k, this transformation:

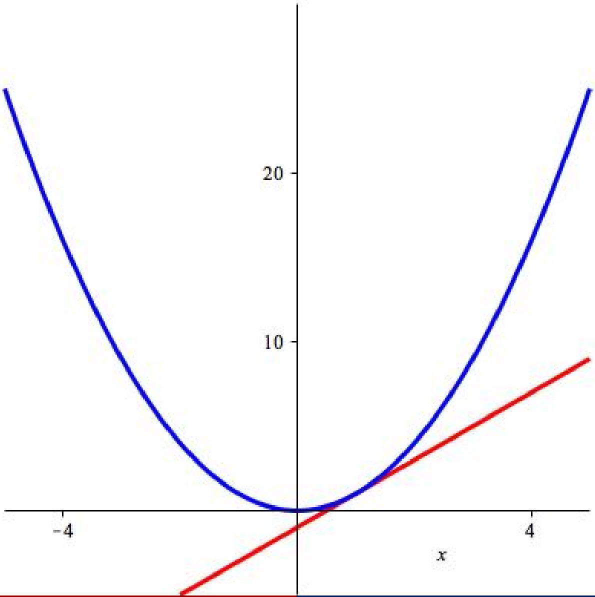
$$x \to kx \qquad y \to k^2 y$$

is a *continuous* symmetry of the equation of the parabola:

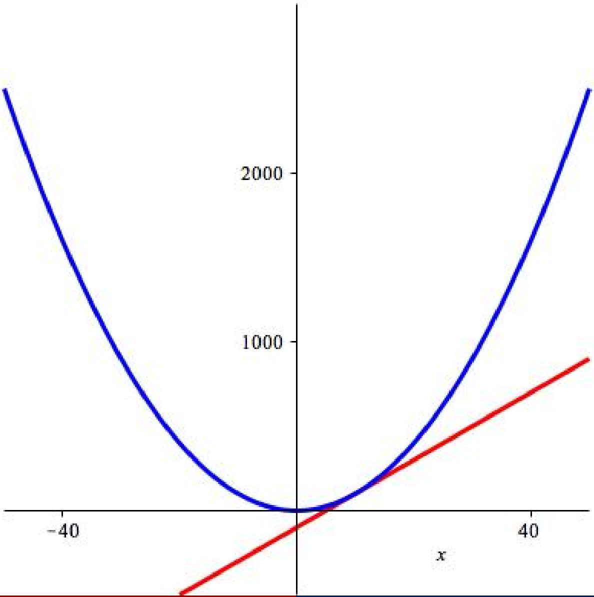
If $y = x^2$, and we set $X = kx$ and $Y = k^2 y$ then $Y = X^2$.

What are we going to do with this symmetry?

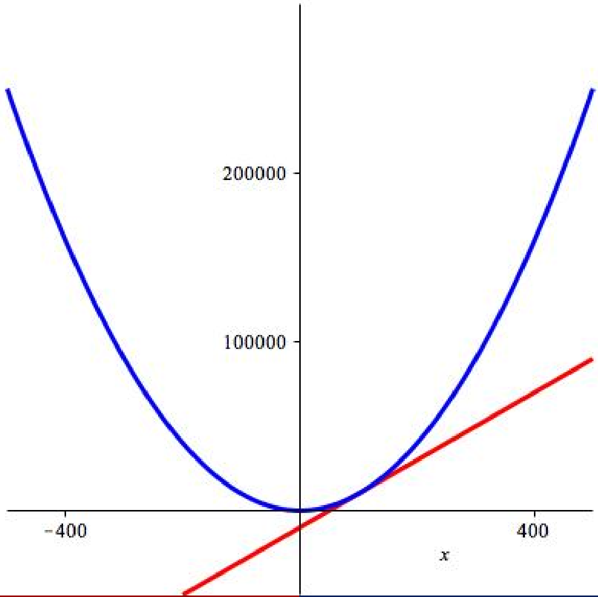
The parabola $y = x^2$ with tangent at $x = 1$



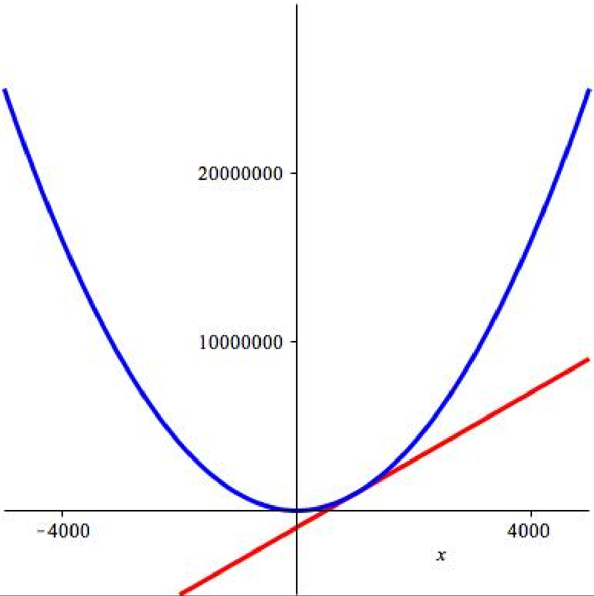
The parabola $y = x^2$ with tangent at $x = 10$



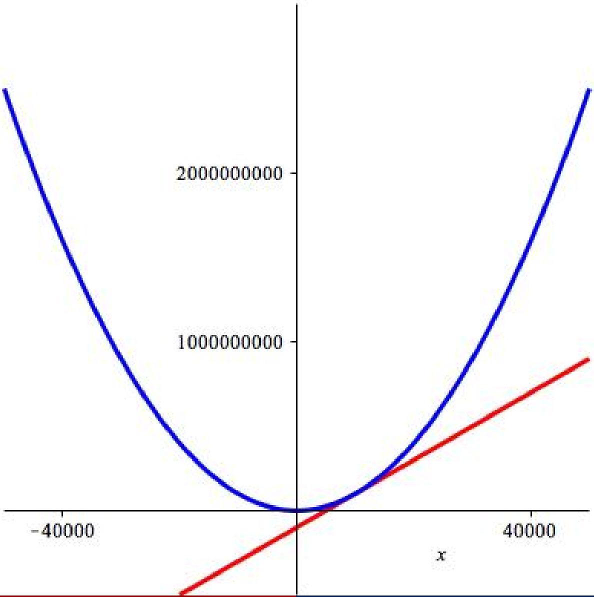
The parabola $y = x^2$ with tangent at $x = 100$



The parabola $y = x^2$ with tangent at $x = 1000$



The parabola $y = x^2$ with tangent at $x = 10000$



Computations

If the tangent line at $(1, 1)$ is

$$y - 1 = m_1(x - 1)$$

and if $Y = k^2 y$ and $X = kx$ then

$$\begin{aligned} Y - k^2 &= m_k(X - k) \\ k^2(y - 1) &= m_k(kx - k) \\ k^2(y - 1) &= km_1(kx - k) \end{aligned}$$

Therefore the slope at k is k times the slope at 1, in other words the derivative of x^2 is a constant times x

Other powers

More generally, the graph of $y = x^p$ is invariant with respect to the transformation

$$x \to kx \qquad y \to k^p y$$

so we can run the computation again:

If the tangent line at 1 is

$$y - 1 = m_1(x - 1)$$

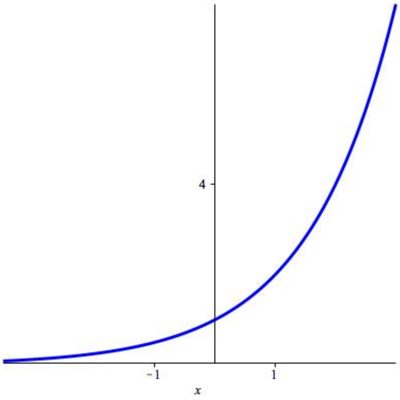
and if $Y = k^p y$ and $X = kx$ then

$$\begin{aligned} Y - k^p &= m_k(X - k) \\ k^p(y-1) &= m_k(kx - k) \\ k^p(y-1) &= k^{p-1} m(kx - k) \end{aligned}$$

Therefore the slope at k is k^{p-1} times the slope at 1, or the derivative of x^p is a constant times x^{p-1}.

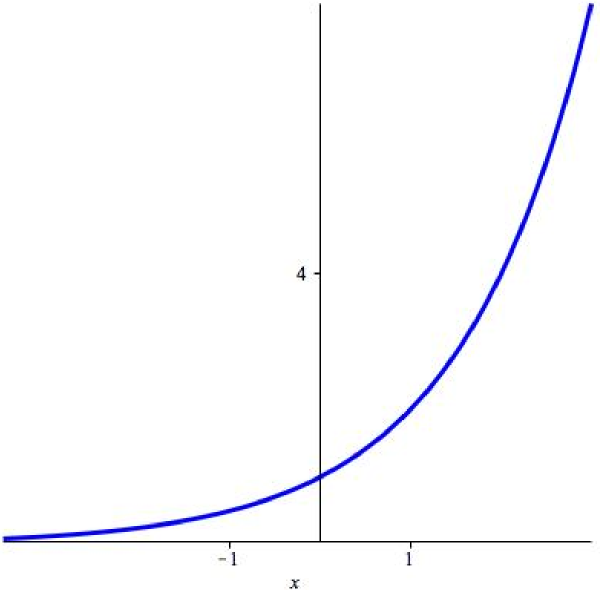
What about exponential functions?

What kind of symmetry does the graph of $y = 2^x$ have?



We'll explore this on the next few slides.

The graph of $y = 2^x$



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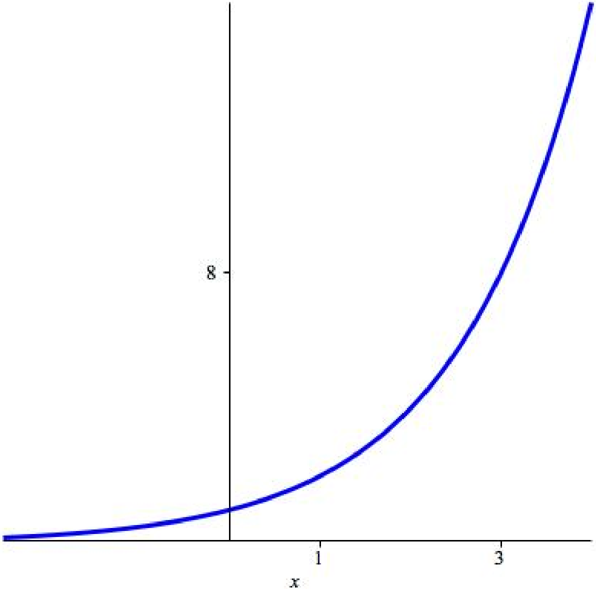

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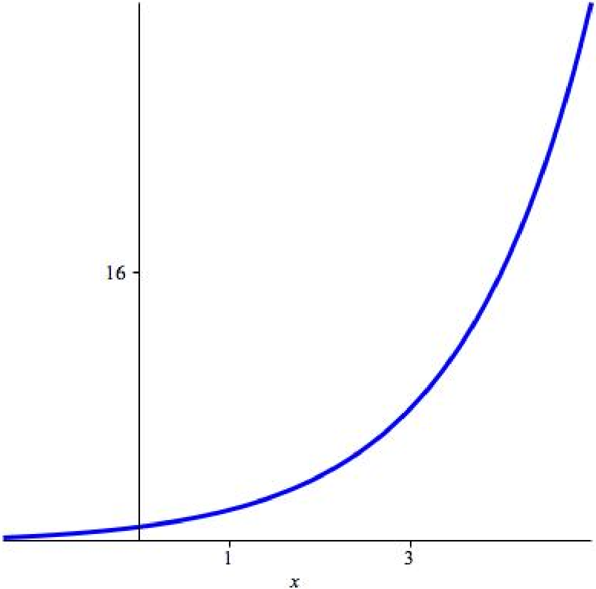

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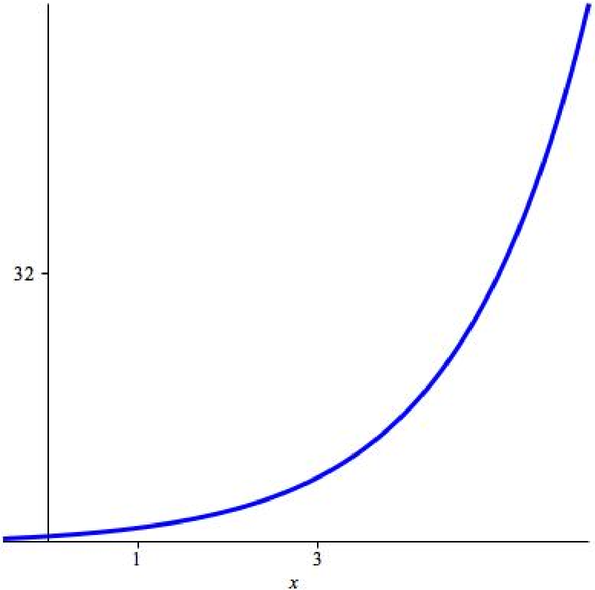

What changed this time?

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The scale on the y-axis changed as the position (but not the scale) of the x-axis changed.

How do we describe this symmetry?

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The scale on the y-axis changed as the position (but not the scale) of the x-axis changed.

How do we describe this symmetry?

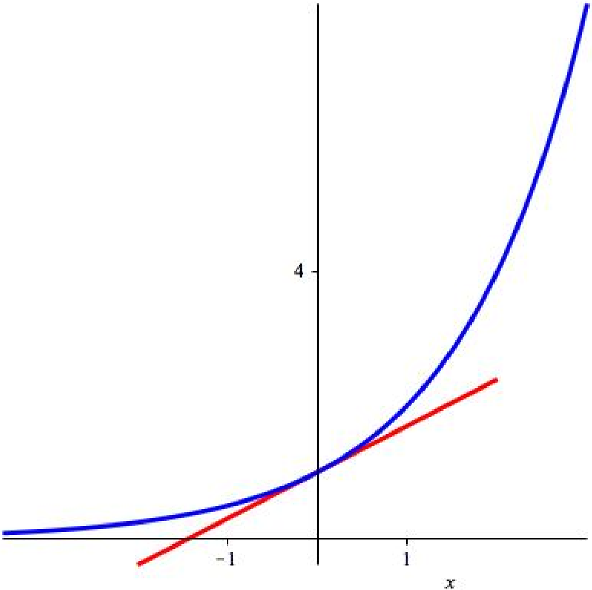
If we add k to the x values then the y values get multiplied by 2^k. So our symmetry transformation is

$$x \to x + k \qquad y \to 2^k y$$

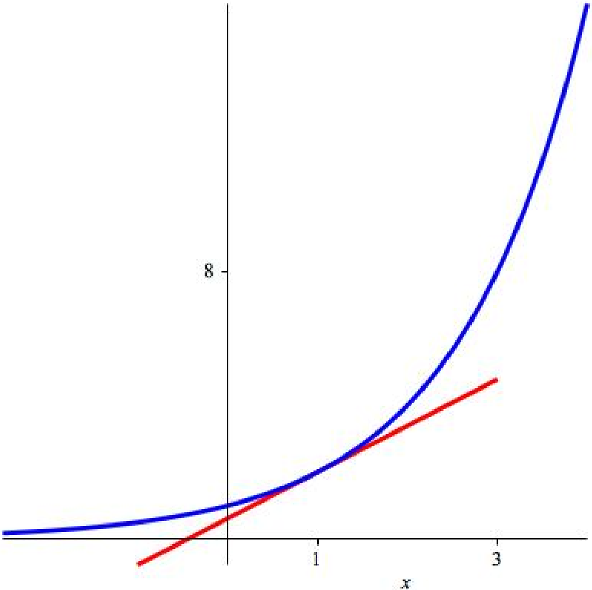
How can we use this to get information about the derivative of $y = 2^x$?

More pictures are in order!

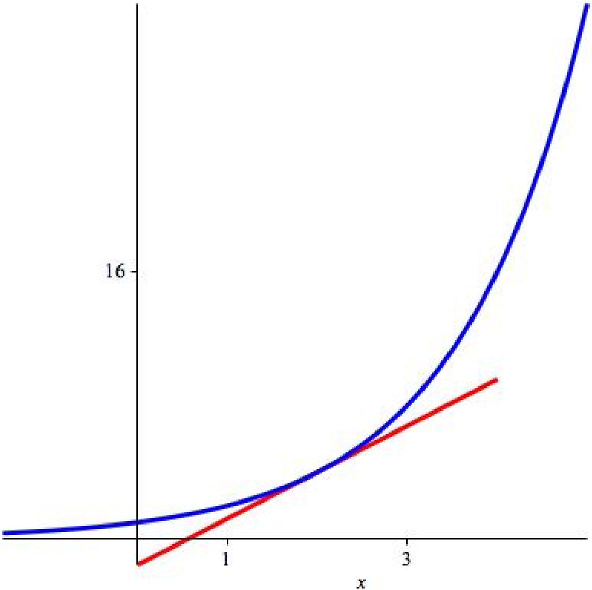
The graph of $y = 2^x$ with tangent line at $x = 0$



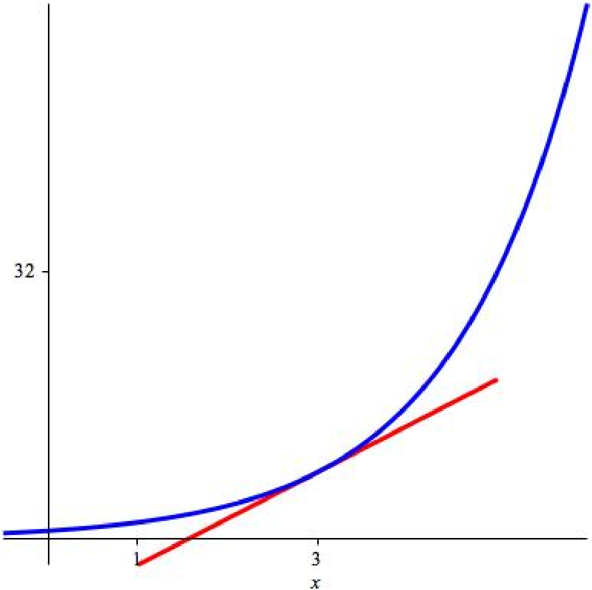
The graph of $y = 2^x$ with tangent line at $x = 1$



The graph of $y = 2^x$ with tangent line at $x = 2$



The graph of $y = 2^x$ with tangent line at $x = 3$



Computations

Well, if the tangent line at $x = 0$ is

$$y - 1 = m_0(x - 0)$$

and if $Y = 2^k y$ and $X = x + k$ then the tangent line at $(k, 2^k)$ is

$$\begin{aligned} Y - 2^k &= m_k(X - k) \\ 2^k y - 2^k &= m_k((x + k) - k) \\ 2^k(y - 1) &= m_k x \end{aligned}$$

Therefore the slope at $x = k$ is 2^k times the slope at $x = 0$.
What is the slope at $x = 0$?

Computations

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and if $Y = 2^k y$ and $X = x + k$ then the tangent line at $(k, 2^k)$ is

$$\begin{aligned} Y - 2^k &= m_k(X - k) \\ 2^k y - 2^k &= m_k((x + k) - k) \\ 2^k(y - 1) &= m_k x \end{aligned}$$

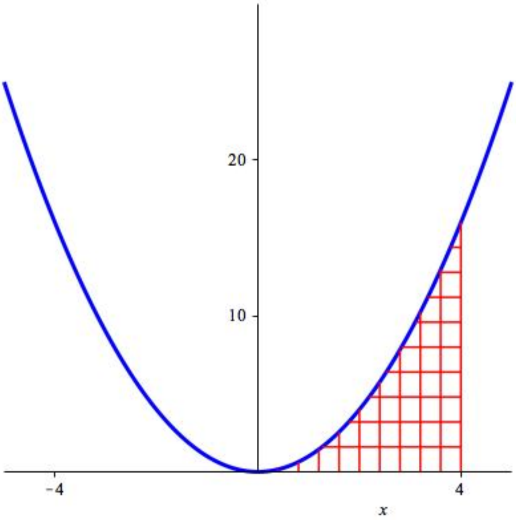
Therefore the slope at $x = k$ is 2^k times the slope at $x = 0$.
What is the slope at $x = 0$?
Clearly this will work for all exponential functions.

Integrals, too!

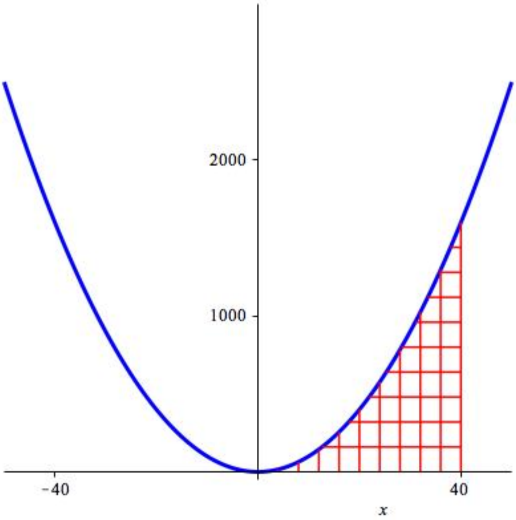
Even without the fundamental theorem, we can learn that the integral of x^p is a constant times x^{p+1} by using our symmetry transformations!

Once again, we'll draw a few pictures for $y = x^2$ to get the idea.

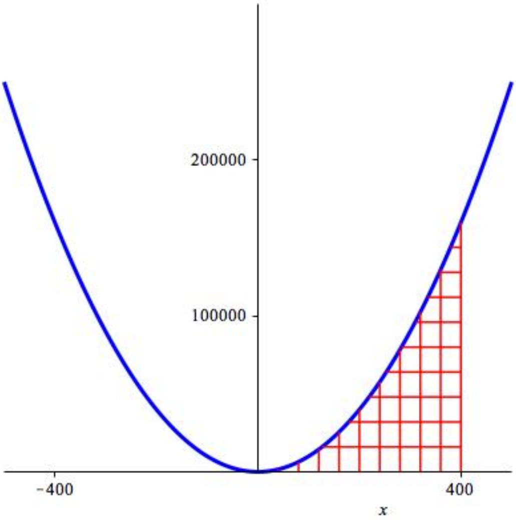
$\int_0^4 x^2\,dx$



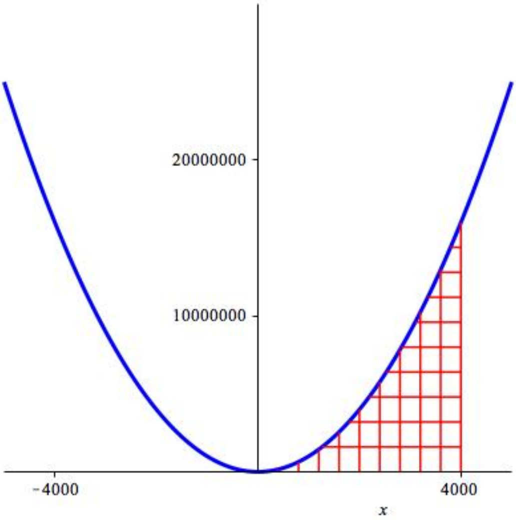
$\int_0^{40} x^2\,dx$



$$\int_0^{400} x^2\, dx$$



$\int_0^{4000} x^2\,dx$



How does the area scale?

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When we multiply x by k and y by k^2 the areas of the boxes (or the terms in the Riemann sum, etc) get multiplied by k^3. We conclude that

$$\int_0^k x^2\,dx = k^3 \int_0^1 x^2\,dx.$$

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When we multiply x by k and y by k^2 the areas of the boxes (or the terms in the Riemann sum, etc) get multiplied by k^3. We conclude that

$$\int_0^k x^2\,dx = k^3 \int_0^1 x^2\,dx.$$

Similarly, we can conclude that

$$\int_0^k x^p\,dx = k^{p+1} \int_0^1 x^p\,dx.$$

An exercise for you: How would you use this to learn about the integral of a^x?

Scaling invariance and dimensional analysis

Scientists and engineers often use scaling invariance via *dimensional analysis* to obtain the form of the answer to an applied problem and reduce its complexity.

The principal kinds of physical units are *length, mass and time*.

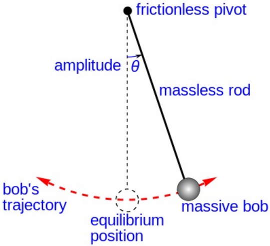
Given a quantity x, we write $[x]$ to denote the *units* of x. For instance for the acceleration of gravity, g, we have

$$[g] = \frac{L}{T^2} = LT^{-2}$$

Note that we don't use any specific system of units, but just specify what kind of unit pertains.

A simple example – the period of the idealized simple pendulum

The idealized simple pendulum has a (point) mass swinging back and forth attached to a (massless, frictionless) string.



What is the *period* P of the pendulum?

What could it depend on?

- the length of the string ℓ, and $[\ell] = L$
- the mass of the bob m, and $[m] = M$
- the acceleration of gravity g, and $[g] = LT^{-2}$.
- and of course $[P] = T$.

Dimensionlessness

Our next task is to create *dimensionless quantities* from ℓ, m, g and P. And we can do this via elementary linear algebra!

The columns of this 3-by-4 matrix represent $[\ell]$, $[m]$, $[g]$ and $[P]$, in terms of L, M and T:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

The rank of this matrix is 3, and its kernel is spanned by

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

So the quantity $P\sqrt{\dfrac{g}{\ell}}$ is dimensionless — it is standard to name such quantities π_1, π_2, etc., so we'll call this one π_1.

Conclusion

It is reasonable to assume that π_1 is a continuous function of whatever parameters we might change about the problem (like the amplitude θ of the swing (which is dimensionless!), the mass, the length, etc..) so we can assume that for small variations in these quantities that

$$P = C\sqrt{\frac{\ell}{g}}$$

which agrees with what we get from the differential equation approximation, namely

$$P = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + o(\theta)\right).$$

A more interesting example – the Trinity test

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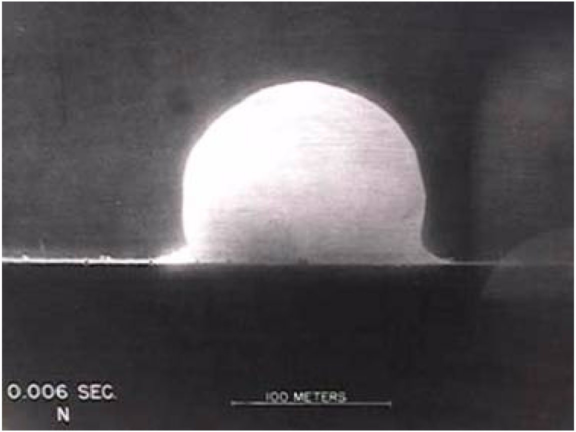

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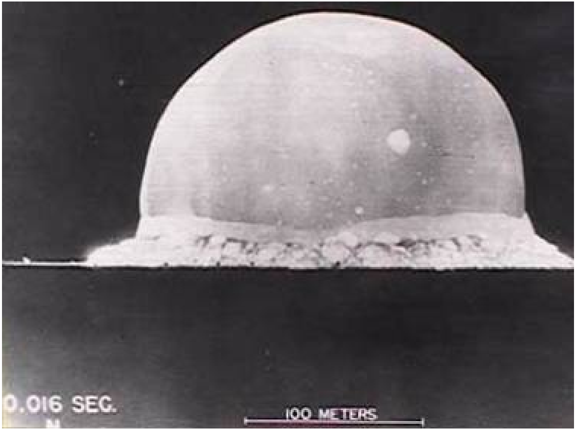

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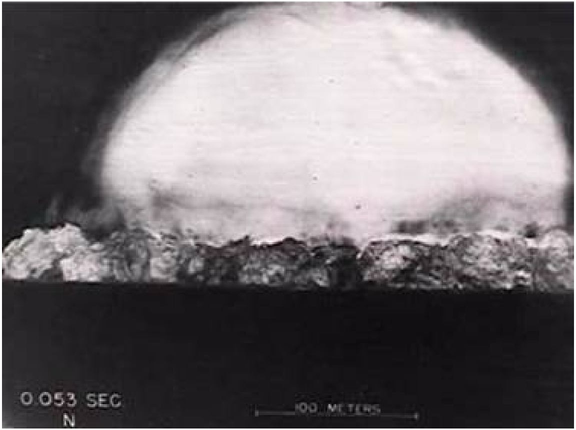

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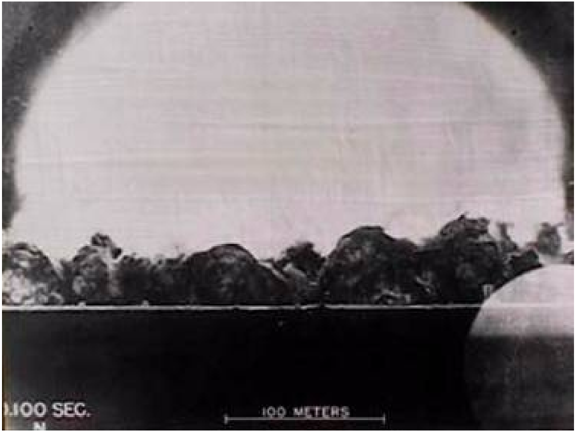

The power of the explosion

Based only on these pictures and dimensional analysis, Sir Geoffrey Taylor was able to calculate the amount of energy released in the explosion, which was still classified at the time (1949).

Taylor asked: *"On what should the size of the fireball depend?"*
And there are four main quantities:

- R – the radius of the fireball (and $[R] = L$)
- E – the energy released by the explosion (and $[E] = ML^2/T^2$)
- t – the time after the explosion (and $[t] = T$)
- ρ – the density of the air (and $[\rho] = M/L^3$)

So we do our linear algebra using R, E, t and ρ

The matrix that represents the units of R, E, t and ρ in terms of L, M, and T this time is

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

Again, this matrix has rank 3 and its kernel is spanned by

$$\begin{bmatrix} 1 \\ -\frac{1}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

Therefore, our dimensionless quantity is

$$\pi_1 = \frac{R\rho^{1/5}}{E^{1/5}T^{2/5}}.$$

Energy estimate

By looking at plots of R vs t for this and several other explosions, and by using established data for small explosives, Taylor determined that π_1 is essentially constant, and is approximately equal to 1. Thus;

$$E = \frac{R^5 \rho}{t^2}$$

From the first picture, the radius of the wave at $t = 0.006$ was about 80 meters. And the density of air is about 1.2 $\mathrm{kg/m^3}$. So

$$E = \frac{80^5 \times 1.2}{0.006^2} \text{ J} \approx 1 \times 10^{14} \text{ J}.$$

And since 1 ton of TNT releases about 4×10^9 J, we have that the Trinity explosion released energy equivalent to about 0.25×10^5, or 25,000 tons of TNT. This was a remarkably accurate estimate!

Scaling in an uncertain environment

Sometimes we can use scaling symmetry even when we're not certain of the exponents! Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(often there is a coefficient k on the right side, but we'll assume it away).

Since u is a function of x and t, we'll consider scaling all three of the variables using the transformation:

$$T = \lambda t \quad X = \lambda^b x \quad U = \lambda^a u.$$

Check the invariance

$$T = \lambda t \quad X = \lambda^b x \quad U = \lambda^a u.$$

If the function $u(x,t)$ is invariant under this transformation, then

$$u(x,t) = \frac{1}{t^a}\, v\left(\frac{x}{t^b}\right).$$

This means that $U(X,T) = u(x,t)$. To see this, calculate:

$$U(X,T) = \lambda^a u(\lambda^b x, \lambda t) = \frac{\lambda^a}{(\lambda t)^a}\, v\left(\frac{\lambda^b x}{(\lambda t)^b}\right)$$
$$= \frac{1}{t^a}\, v\left(\frac{x}{t^b}\right) = u(x,t)$$

So we'll substitute this expression for u into the heat equation.

Into the heat equation

Since $u(x,t) = \frac{1}{t^a}\ v\left(\frac{x}{t^b}\right)$, we have

$$\frac{\partial u}{\partial t} = -\frac{a}{t^{a+1}}\ v\left(\frac{x}{t^b}\right) - \frac{b}{t^{a+1}}\frac{x}{t^b}\ v'\left(\frac{x}{t^b}\right).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{t^{a+2b}}\ v''\left(\frac{x}{t^b}\right).$$

So if we put $b = \frac{1}{2}$, then the heat equation becomes:

$$0 = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\frac{1}{t^{a+1}}\left(av(y) + \frac{y}{2}v'(y) + v''(y)\right)$$

on setting $y = x/t^b$, and we've reduced the partial differential equation to an ordinary differential equation.

Solving the ODE $v'' + \frac{1}{2}yv' + av = 0$

We can solve the ODE if we put $a = \frac{1}{2}$: It becomes

$$(v')' + \frac{1}{2}(yv)' = 0,$$

so $v' + \frac{1}{2}yv$ is a constant q. From physical considerations, we expect v and v' to approach zero as $y \to \pm\infty$, so $q = 0$ and now it's easy to solve and get $v = Ce^{-y^2/4}$

Now, recall that $y = \dfrac{x}{\sqrt{t}}$ and $u = \dfrac{1}{\sqrt{t}}\, v\left(\dfrac{x}{\sqrt{t}}\right)$, so that

$$u = \frac{C}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$$

If $C = 1/(2\sqrt{\pi})$, so that the integral of u from $x = -\infty$ to ∞ is 1 for all t, then this function is the *fundamental solution of the heat equation*. And so it goes. . .