The wave equation on the disk

We've solved the wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

on rectangles. Now we'll consider it on a circular disk $x^2 + y^2 < a^2$. Of course, it's natural to use polar coordinates so we rewrite the wave equation as:

$$u_{tt} = c^2 \left(\frac{1}{r}(ru_r)_r + \frac{1}{r^2} u_{\theta\theta} \right)$$

and solve for u as a function of r, θ and t.

We'll assume homogeneous boundary conditions

$$u(a, \theta, t) = 0$$

and of course that u is periodic with period 2π in θ.

And we'll have the standard initial position and velocity conditions:

$$u(r, \theta, 0) = f(r, \theta) \qquad u_t(r, \theta, 0) = g(r, \theta).$$

To begin the separation of variables process, we'll first separate out the time variable. So we assume

$$u(r, \theta, t) = \varphi(r, \theta) T(t)$$

and transform the wave equation into

$$\varphi T'' = c^2 \left(\frac{1}{r}(r\varphi_r)_r + \frac{1}{r^2} \varphi_{\theta\theta} \right) T$$

and so

$$\frac{T''}{c^2 T} = \frac{\frac{1}{r}(r\varphi_r)_r + \frac{1}{r^2}\varphi_{\theta\theta}}{\varphi} = -\lambda$$

which gives us the equation

$$T'' + \lambda c^2 T = 0$$

for T and (multiplying by $r^2\varphi$)

$$r^2 \varphi_{rr} + r\varphi_r + \varphi_{\theta\theta} + \lambda r^2 \varphi = 0.$$

This is equivalent to the Helmholtz equation (or reduced wave equation) for φ, namely $\Delta\varphi + \lambda\varphi = 0$.

Next, we'll separate variables in the Helmholtz equation, so we assume that $\varphi(r, \theta) = R(r)\Theta(\theta)$ and we get

$$r^2 R''\Theta + rR'\Theta + R\Theta'' + \lambda r^2 R\Theta = 0.$$

We divide by $R\Theta$ and rearrange a bit to get:

$$\frac{r^2R'' + rR' + \lambda r^2 R}{R} = -\frac{\Theta''}{\Theta} = \mu$$

for a constant μ. This gives us the Θ equation

$$\Theta'' + \mu\Theta = 0,$$

and since Θ must be periodic with period 2π, we get that $\mu = 0, 1, 4, \ldots, n^2, \ldots$ and

$$\Theta = a_n \cos n\theta + b_n \sin n\theta.$$

Since we know $\mu = n^2$, we get that the R equation is:

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0$$

and the boundary conditions for R are $R(a) = 0$ and $R(0)$ is bounded.

If $\lambda > 0$, we can make a change of variables in the R equation that will eliminate λ from the equation. Let $x = \sqrt{\lambda}r$. Then

$$\frac{dR}{dr} = \frac{dR}{dx}\frac{dx}{dr} = \sqrt{\lambda}\frac{dR}{dx} \quad \text{and} \quad \frac{d^2R}{dr^2} = \lambda\frac{d^2R}{dx^2}$$

and the R equation becomes

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - n^2)R = x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0.$$

The equation

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0$$

is called *Bessel's equation of order n.*

Solving Bessel's equation.

We're going to solve Bessel's equation using power series. But because of the coefficient x^2 in front of d^2R/dx^2 and x in front of dR/dx (which are zero when $x = 0$), we can't assume that R has a standard Maclaurin series. Rather, we assume that R is some power of x times a Maclaurin series, so

$$R(x) = x^s \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+s}.$$

We assume s can be chosen so that $a_0 \neq 0$. We then have

$$R'(x) = \sum_{k=0}^{\infty}(k+s)a_k x^{k+s-1} \quad \text{and} \quad R''(x) = \sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-2}$$

We substitute this into Bessel's equation and obtain:

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s} + (k+s)a_k x^{k+s} + a_k x^{k+s+2} - n^2 a_k x^{k+s} = 0.$$

Simplify a bit to obtain

$$\sum_{k=0}^{\infty}((k+s)^2 - n^2)a_k x^{k+s} + \sum_{k=2}^{\infty} a_{k-2}x^{k+s} = 0$$

or

$$(s^2-n^2)a_0x^s + ((s+1)^2-n^2)a_1x^{s+1} + \sum_{k=2}^{\infty}\left(((k+s)^2-n^2)a_k + a_{k-2}\right)x^{k+s} = 0.$$

Since we need every coefficient to be zero, we get from the first two terms that $s = \pm n$ (since we assume $a_0 \neq 0$), and then we have $a_1 = 0$. Because we want R bounded when $x = 0$, we'll assume that $s = +n$. Then we have the *recurrence relation*

$$((k+n)^2 - n^2)a_k = -a_{k-2}$$

or

$$a_k = -\frac{a_{k-2}}{k(2n+k)}.$$

From this we see immediately that $a_3 = a_5 = a_7 = \cdots = 0$. For the even coefficients we have

$$\begin{aligned}
a_2 &= -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2\cdot 1\cdot(n+1)}\\
a_4 &= -\frac{a_2}{4(2n+4)} = -\frac{a_2}{2^2\cdot 2\cdot(n+2)} = \frac{a_0}{2^4\cdot 2!\cdot(n+1)(n+2)}\\
a_6 &= -\frac{a_4}{6(2n+6)} = -\frac{a_4}{2^2\cdot 3\cdot(n+3)} = -\frac{a_0}{2^6\cdot 3!\cdot(n+1)(n+2)(n+3)}
\end{aligned}$$

and so forth. If we set

$$a_0 = \frac{1}{2^n n!}$$

then we'll have

$$a_{2k} = \frac{(-1)^k}{2^{n+2k}k!(k+n)!}$$

and so

$$R(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

This last function is called the *Bessel function of the first kind of order* n and is usually denoted $J_n(x)$. This definition can work for all $n \geq 0$, whether or not n is an integer, provided we come up with a definition for $(k+n)!$ when n is not an integer.

A few observations: J_n is an even function if n is an even number, and is an odd function if n is an odd number. $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geq 1$. You could write out the series for J_0 as

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

which looks a little like the series for $\cos x$.

In the homework from a month or so ago, you showed that the Bessel functions have infinitely many zeroes that are spaced about π apart.

You can prove the following formulas using the series:

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x) \quad \text{for} \quad n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

and using these you can show

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

Orthogonality

We know that $J_n(x)$ has infinitely many positive zeros, and we will denote these by z_{nm} for $m = 1, 2, 3, \ldots$. In order to expand a function $f(x)$ in terms of a fixed Bessel function, i.e.,

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(z_{nm} x)$$

we need orthogonality relations. Here they are: If $m \neq k$ then

$$\int_0^1 x J_n(z_{nm} x) J_n(z_{nk} x) \, dx = 0$$

and

$$\int_0^1 x (J_n(z_{nm} x))^2 \, dx = \frac{1}{2} J_{n+1}(z_{nm})^2.$$

In other words, the functions $J_n(z_{nm}x)$ are orthogonal on the interval $0 \le x \le 1$ with respect to the *weight function* x.

To prove these, we begin by writing Bessel's equation of order n as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0,$$

and we know that a solution of this is $y = J_n(x)$. We can use a change of variables similar to the one in the middle of page 2 to show that if α is a positive constant, then the function $u(x) = J_n(\alpha x)$ is a solution of

$$u'' + \frac{1}{x}u' + \left(\alpha^2 - \frac{n^2}{x^2}\right) u = 0.$$

Likewise, if β is another positive constant then $v(x) = J_n(\beta x)$ is a solution of

$$v'' + \frac{1}{x}v' + \left(\beta^2 - \frac{n^2}{x^2}\right) v = 0.$$

Here comes the Wronskian! Multiply the u equation by xv and the v equation by xu and subtract them to obtain:

$$\frac{d}{dx}\left(x(u'v - v'u)\right) = (\beta^2 - \alpha^2)xuv,$$

then integrate from 0 to 1 to get

$$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \left[x(J_n'(\alpha x)J_n(\beta x) - J_n'(\beta_x)J_n(\alpha x)\right]\Big|_{x=0}^{x=1}.$$

So if α and β are distinct positive zeros of $J_n(x)$, say $\alpha = z_{nm}$ and $\beta = z_{nk}$, then

$$(z_{nm}^2 - z_{nk}^2)\int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0$$

which proves the orthogonality of $J_n(z_{nm}x)$ and $J_n(z_{nk}x)$ on $0 \le x \le 1$ with respect to the weight function x.

We'll leave the integral of $x(J_n(z_{nm}x))^2$ as an exercise (you start by multiplying the u equation by $2x^2u'$ and integrating).

Starting from these orthogonality relations, we can derive the *Fourier-Bessel series expansion* in the same way we did for ordinary Fourier series:

For a piecewise smooth function $f(x)$ on the interval $0 \le x \le 1$, we can express

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(z_{nm}x)$$

where

$$a_m = \frac{2}{J_{n+1}(z_{nm})^2} \int_0^1 x f(x) J_n(z_{nm} x)\, dx.$$

The series will converge to $f(x)$ wherever f is continuous, and to the average of the left and right limits of f at points where f has a jump discontinuity.

Back to the wave equation

Where were we? We had separated variables in

$$u_{tt} = c^2 \left(\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} \right)$$

to obtain:

$$T'' + \lambda c^2 T = 0,$$

$$\Theta'' + n^2 \Theta = 0$$

(where we know that $n = 0, 1, 2, \ldots$), and

$$r^2 R'' + r R' + (\lambda r^2 - n^2) R = 0$$

(where we need $R(a) = 0$, where a is the radius of our disk).

We now know that if we set $x = \sqrt{\lambda}\, r$, then the R equation becomes Bessel's equation

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + (x^2 - n^2) R = 0$$

which has solution $c J_n(x) = c J_n(\sqrt{\lambda}\, r)$.

To satisfy the boundary condition we need $J_n(\sqrt{\lambda}\, a) = 0$, so we'll set

$$\lambda_{nm} = \left(\frac{z_{nm}}{a} \right)^2$$

for $n \geq 0$ and $m \geq 1$. Our corresponding eigenfunctions are thus

$$R_{nm}(r) = J_n\left(\frac{z_{nm}}{a} r \right).$$

We know that the solutions of the Θ equation are cosines and sines of $n\theta$. And now we can solve the T equation because we know what the λs are:

$$T_{nm}(t) = A \cos(\sqrt{\lambda_{nm}} ct) + B \sin(\sqrt{\lambda_{nm}} ct).$$

We put it all together and get

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\cos(\sqrt{\lambda_{nm}}ct)\left(a_{nm}\cos n\theta + b_{nm}\sin n\theta\right)$$
$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sin(\sqrt{\lambda_{nm}}ct)\left(c_{nm}\cos n\theta + d_{nm}\sin n\theta\right)$$

The first double sum will take the initial position $u(r,\theta,0) = f(r,\theta)$ into account, and the second double sum will take the initial velocity into account.

To calculate the coefficients (we'll just do the a_{mn} and b_{mn} and leave the others as an exercise), note that we need

$$f(r,\theta) = u(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)(a_{nm}\cos n\theta + b_{nm}\sin n\theta)$$

If we view θ as the variable and r as constant for the moment, this becomes an ordinary Fourier series for $f(r,\theta)$, so we have

$$\sum_{m=1}^{\infty} a_{0m}J_0(\sqrt{\lambda_{0m}}r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\,d\theta \quad \text{for } n=0,$$
$$\sum_{m=1}^{\infty} a_{nm}J_n(\sqrt{\lambda_{nm}}r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos n\theta\,d\theta \quad \text{for } n\geq 1,$$
$$\sum_{m=1}^{\infty} b_{nm}J_n(\sqrt{\lambda_{nm}}r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin n\theta\,d\theta \quad \text{for } n\geq 1.$$

But the left sides of these are Fourier-Bessel series, so using the results of the previous section we finally obtain the coefficients:

$$a_{0m} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_0(\sqrt{\lambda_{0m}}r)\,dr\,d\theta}{\int_0^a rJ_0(\sqrt{\lambda_{0m}}\,r)^2\,dr} \quad \text{for } n=0, m\geq 1$$

$$a_{nm} = \frac{\frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n(\sqrt{\lambda_{nm}}r)\cos n\theta\,dr\,d\theta}{\int_0^a rJ_n(\sqrt{\lambda_{nm}}\,r)^2\,dr} \quad \text{for } n\geq 1, m\geq 1$$

$$b_{nm} = \frac{\frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^a rf(r,\theta)J_n(\sqrt{\lambda_{nm}}r)\sin n\theta\,dr\,d\theta}{\int_0^a rJ_n(\sqrt{\lambda_{nm}}\,r)^2\,dr} \quad \text{for } n\geq 1, m\geq 1$$

and the denominators are given by

$$\int_0^a r J_n(\sqrt{\lambda_{nm}}\, r)^2\, dr = \int_0^a r J_n\left(\frac{z_{nm}}{a} r\right)^2 dr = \frac{a^2}{2} J_{n+1}(z_{nm})^2.$$

Exercises

1. Prove formulas (1)–(6) concerning Bessel functions.
2. Multiply the Bessel equation
$$u'' + \frac{1}{x}u' + \left(\alpha^2 - \frac{n^2}{x^2}\right) u = 0$$
by $2x^2 u'$ and integrate from 0 to 1 and show that
$$\int_0^1 x J_n(\alpha x)^2\, dx = \frac{1}{2} J_n'(\alpha)^2 + \frac{1}{2}\left(1 - \frac{n^2}{\alpha^2}\right) J_n(\alpha)^2.$$
Then put $\alpha = z_{nm}$ and conclude (using formula (4)) that
$$\int_0^1 x J_n(z_{nm} x)^2\, dx = \frac{1}{2} J_n'(z_{nm})^2 = \frac{1}{2} J_{n+1}(z_{nm})^2.$$
3. Calculate the coefficients c_{nm} and d_{nm} in the solution of the wave equation.
4. Prove the formula
$$\int_0^a x^{n+1} J_n\left(\frac{\alpha x}{a}\right) dx = \frac{a^{n+2}}{\alpha} J_{n+1}(\alpha)$$
and use it to solve the problem (with circular symmetry, so there's no dependence on θ):
$$\frac{\partial^2 u}{\partial t^2} = 16\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 < r < 1,\ t > 0$$
with boundary condition $u(1,t) = 0$ and initial conditions $u(r,0) = 1 - r^2$ and $u_t(r,0) = 1$.
(*Hint*: I think the answer is
$$u(r,t) = \sum_{m=1}^{\infty} J_0(z_m r)\left[\frac{8}{z_m^3 J_1(z_m)}\cos(4z_m t) + \frac{1}{2z_m^2 J_1(z_m)}\sin(4z_m t)\right].$$
where z_m is the mth positive zero of the Bessel function $J_0(x)$. You'll need to use identities (1) and (6) and integration by parts to get it into this form.)