8. Fourier transforms

We begin with a little motivation – the Fourier series we have been studying

\[ f = \sum \hat{f}(n)e_n = \sum \hat{f}(n)e^{2\pi inx} \]
on the circle \( S^1 \) may be thought of as an expansion of a periodic function of period 1 into harmonics of the same period. Of course, the choice of period is just a matter of convenience. For functions of arbitrary period \( T > 0 \), the appropriate harmonics (orthonormal family) are

\[ e_n(x) = \frac{e^{2\pi inx/T}}{\sqrt{T}}, \]

and any “nice” function \( f(x) \) of period \( T \) can be expanded as

\[ f(x) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{T} \int_{-T/2}^{T/2} f(y)e^{-2\pi iny/T} dy \right) e^{2\pi inx/T}. \]

If you squint at this a little, it will turn into the Fourier transform: The idea is that the right-hand side is like a Riemann sum over a subdivision with spacing \( 1/T \) and with any luck, it should approximate the integral

\[ f(x) = \int_{-\infty}^{\infty} f(y)e^{-2\pi i\xi y} dy e^{2\pi i\xi x} d\xi \]
as \( T \to \infty \). This actually does not make sense for a periodic function \( f \), since the integral won’t converge, but it does suggest that something can be done to recover a nice function \( f \) from its Fourier transform:

\[ \hat{f}(\xi) = \int_{-\infty}^{\infty} f(y)e^{-2\pi i\xi y} dy \]

via the inverse Fourier transform

\[ \hat{f}(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi x} d\xi. \]

In this section of the notes, we’ll put this heuristic discussion on a solid mathematical foundation, for functions from \( S(\mathbb{R}) \), \( L^2(\mathbb{R}) \), and \( L^1(\mathbb{R}) \). The new function space, \( S(\mathbb{R}) \), is the set of infinitely differentiable rapidly decreasing functions on \( \mathbb{R} \). “Infinitely differentiable” means that \( f \in C^\infty(\mathbb{R}) \), and “rapidly decreasing” applies to \( f, f', f'', \ldots \), and means that \( x^p D^q f \) tends to zero as \( |x| \to \infty \) for every non-negative integer value of \( p \) and \( q \). Here, \( D \) stands for differentiation, \( Df = f' \).

Exercise 1: Show that \( e^{-x^2} \) belongs to \( S(\mathbb{R}) \), but \( 1/(1+x^2) \) does not, nor does \( e^{-|x|} \), but for a different reason.

Convention: Unless otherwise specified, \( \int \) stands for \( \int_{-\infty}^{\infty} f(x)dx \) in what follows.
$L^2(\mathbb{R})$ is the space of measurable functions $f$ with
\[ \|f\|_2 = \left( \int f^2 \right)^{1/2} < \infty. \]
$L^1(\mathbb{R})$ is the space of summable functions with
\[ \|f\|_1 = \int |f| < \infty. \]
$L^1(\mathbb{R})$ is an algebra with respect to the “convolution” product
\[ f * g = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \]
just as for the circle.

**Exercise 2**: Check that the convolution is associative and commutative, and verify the bound $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

It is the case that $S(\mathbb{R})$ is dense in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

**Exercise 3**: Check that translation is continuous in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, in other words, verify that in either space $f_y(x) = f(x+y)$ is close to $f$ for small $|y|$.

**Fourier Transforms for $S(\mathbb{R})$**

To define the Fourier transform for functions $f$ in $L^2(\mathbb{R})$, it is not sufficient just to write down the formal integral for $\hat{f}$, since the integrand is not necessarily summable. This trouble does not arise for $f \in L^1(\mathbb{R})$, but then $\hat{f}$ might not be summable, so you might have trouble with the inverse Fourier transform. For example, the indicator function of the interval $-1 \leq x \leq 1$ is summable, but
\[ \int_{-1}^{1} e^{-2\pi i x}dx = \frac{\sin 2\pi \xi}{\pi \xi} \]
is not. On the other hand, the Schwarz class $S(\mathbb{R})$ does not have any problems of this kind, so we’ll begin there.

**Theorem**:

(a) The Fourier transform maps $S(\mathbb{R})$ to itself.

(b) For such functions the inverse transform does what it should:
\[ \hat{\hat{f}} = f \]

(c) There is a Plancherel identity:
\[ \|f\|_2 = \|\hat{f}\|_2 \]
The proof consists in justifying the formal development above (we’ll do another proof later):

Pick a function \( f \in S(\mathbb{R}) \), and use integration by parts to show
\[
\hat{f}' = \int f'(x)e^{-2\pi i \xi x} \, dx = -\int f(x)(e^{-2\pi i \xi x})' \, dx = 2\pi i \xi \hat{f}.
\]
Similarly, using the rapid decrease of \( f \),
\[
-2\pi i \xi f = (\hat{f})'.
\]
By induction, we can show that
\[
(2\pi i \xi)^p D^q \hat{f} = D^p(-2\pi i \xi)^q f,
\]
for any non-negative integers \( p \) and \( q \). Therefore,
\[
|\xi|^p |D^q \hat{f}| \leq (2\pi)^{q-p} \|D^p x^q f\|_1 < \infty,
\]
so that \( \hat{f} \) also belongs to \( S(\mathbb{R}) \).

Now we’ll prove the theorem for functions with “compact support”, i.e., functions that are zero outside some interval \([-T/2, T/2]\). If \( f \) is such a function, we can regard it as an infinitely differentiable function on the circle of length \( T \). Since \( f \) is in this sense a \( C^\infty \) function on the circle, its Fourier series of period \( T \) will be rapidly convergent:
\[
f(x) = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi inx/T}}{T} \int_{-T/2}^{T/2} f(y) e^{-2\pi i ny/T} \, dy
\]
\[
= \sum_{n=-\infty}^{\infty} \frac{\hat{f}(n/T)}{T} e^{2\pi inx/T}
\]
(the \( \hat{f} \) here is the actual Fourier transform, because \( f \) has compact support). But this is just a Riemann sum approximating the integral
\[
\hat{f}(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi,
\]
and so in order to prove that
\[
\hat{f} = f
\]
for compactly supported functions \( f \), you only have to check that the sum converges to the integral as \( T \to \infty \). The same line of reasoning leads from the formula
\[
\|f\|_2^2 = \int_{-T/2}^{T/2} |f|^2 = \sum_{n=-\infty}^{\infty} \frac{|\hat{f}(n/T)|^2}{T}
\]
to the Plancharel identity:

\[ \|f\|_2 = \|\hat{f}\|_2 = \left( \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}. \]

**Exercise 4**: Complete the proof of the two arguments sketched above, putting in all the necessary estimates. (Hint: It might help to use the fact that \( \hat{f} \in S(\mathbb{R}) \) to bound \( |\hat{f}| \) by a constant multiple of \( 1/(1 + \xi^2) \)).

To deal with the other functions in \( S(\mathbb{R}) \), those that do not have compact support, begin with a \( C^\infty \) function \( \varphi(x) \) that is identically equal to 1 on the interval \([-1/2, 1/2]\), is between 0 and 1 for all \( x \), and is identically zero for \( x > 1 \) and for \( x < -1 \). For a general \( f \in S(\mathbb{R}) \), put \( f_n = f(x) \times \varphi(x/n) \). Then

\[ \|\hat{f} - \hat{f}_n\|_\infty \leq \|f - f_n\|_1 \leq \int_{|x|>n/2} |f(x)| dx, \]

which tends to zero as \( n \to \infty \).

We want to prove that \( \hat{f}_n \) converges to \( \hat{f} \), and use various limit theorems to prove that the Fourier inversion formula holds for \( f \) as well as for \( f_n \). To prepare to use the dominated convergence theorem for this purpose, note that \( \hat{f}_n \) and also \( \hat{f} \) are dominated by the following function, which is in both \( L^1(\mathbb{R}) \) and in \( L^2(\mathbb{R}) \):

\[ Q(\xi) = \begin{cases} 
\|f\|_1 & |\xi| \leq 1 \\
\frac{K}{\xi^2} & |\xi| > 1 
\end{cases} \]

where \( K \) is a constant – we prove that such a constant exists by the following estimate:

\[ |\hat{f}_n(\xi)| = \frac{1}{4\pi^2\xi^2} |\hat{f}'(\xi)| \]
\[ \leq \frac{1}{4\pi^2\xi^2} \|f''_n\|_1 \]
\[ \leq \frac{1}{4\pi^2\xi^2} \left( \|f''(x)\varphi(x/n)\|_1 + \frac{2}{n} f'(x)\varphi(x/n) + \frac{1}{n^2} f(x)\varphi''(x/n) \right) \]
\[ \leq \frac{1}{4\pi^2\xi^2} (\|f''\|_1 \|\varphi\|_\infty + \frac{2}{n} \|f'\|_1 \|\varphi\|_\infty + \frac{1}{n^2} \|f\| \|\varphi''\|_\infty). \]

Clearly the last line is bounded by a constant multiple of \( 1/\xi^2 \), independent of \( n \geq 1 \).

Since the sequence \( \{\hat{f}_n\} \) is dominated in \( L^1(\mathbb{R}) \), we can verify the Fourier inversion formula:

\[ f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \int \hat{f}_n(\xi)e^{2\pi i \xi x} d\xi = \int \hat{f}(\xi)e^{2\pi i \xi x} d\xi = \hat{\hat{f}}(x). \]
And since the same sequence is dominated in $L^2(\mathbb{R})$, we also have the Plancheral identity:

$$\|f\|_2 = \lim_{n \to \infty} \|f_n\|_2 = \lim_{n \to \infty} \|\hat{f}_n\|_2 = \|\hat{f}\|_2.$$

The rest is pretty easy: We already know that the Fourier transform maps $S(\mathbb{R})$ into itself, and to see that the map is onto, you only have to use the identity:

$$f = \hat{\hat{f}} = \int \hat{f}(-\xi)e^{-2\pi i \xi x} d\xi,$$

after the change of variables $\xi \to -\xi$, which displays the general function $f \in S(\mathbb{R})$ as the Fourier transform of $\hat{f}(-\xi)$.

**Fourier Transforms for $L^2$ Functions**

Because $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we can extend the Fourier transform from $S(\mathbb{R})$ to $L^2(\mathbb{R})$ without too much effort: For $f \in L^2(\mathbb{R})$, choose a sequence of functions $\{f_n\}$ in $S(\mathbb{R})$ with $\lim \|f_n - f\|_2 = 0$. Because we know the Plancherel identity is true in $S(\mathbb{R})$,

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 = \|\hat{f}_n - \hat{f}_m\|_2 \leq \|f_n - f\|_2 + \|f - f_m\|_2,$$

so we can define $\hat{f}$ as the limit (in the sense of $L^2$) of the sequence $\{\hat{f}_n\}$, and be assured that the limit really exists.

**Exercise 5:** Show that $\hat{f}$ is well-defined, meaning that it depends only on $f$ and not on the particular sequence of $S(\mathbb{R})$ functions used to approximate $f$. Also check that, with this definition, the Fourier transform is a linear map.

We also get the Plancherel identity in $L^2$:

$$\|f\|_2 = \lim_{n \to \infty} \|f_n\|_2 = \lim_{n \to \infty} \|\hat{f}_n\|_2 = \|\hat{f}\|_2,$$

and so we have that the Fourier transform is a one-to-one, length preserving map of $L^2(\mathbb{R})$ into itself. But the definition of the inverse Fourier transform may be extended from $S(\mathbb{R})$ to $L^2(\mathbb{R})$ in the same way, and so we obtain the inversion formula on $L^2$ with no extra effort:

$$\hat{f} = \lim_{n \to \infty} \hat{f}_n = \lim_{n \to \infty} \hat{f}_n = \lim_{n \to \infty} \hat{f}_n = \lim_{n \to \infty} f_n = f = \hat{f}.$$

In summary, both the Fourier and inverse Fourier transforms are isomorphisms of $L^2(\mathbb{R})$ onto itself, and (naturally) are inverses of one another.

A more concrete way to express the Fourier transform on $L^2$ is

$$\hat{f} = \lim_{b \to \infty} \lim_{a \to -\infty} \int_a^b f(x)e^{-2\pi i \xi x} dx,$$
and for the inverse transform we have

\[ f = \hat{f} = \lim_{a \to -\infty} \lim_{b \to \infty} \int_{a}^{b} \hat{f}(\xi)e^{2\pi i \xi x} d\xi, \]

where the limits are to be understood in the sense of \( L^2(\mathbb{R}) \). To prove this, let \( f_{ab} \) be the product of \( f \) with the indicator function of the interval \([a, b] \). Then

\[ \| \hat{f} - \hat{f}_{ab} \|_2 = \| \hat{f} - \hat{f}_{ab} \|_2 = \| f - f_{ab} \|_2 \]

tends to zero as \( a \to -\infty \) and \( b \to \infty \). To finish the proof, you only have to check that

\[ \hat{f}_{ab} = \int_{a}^{b} f(x)e^{-2\pi i \xi x} dx. \]

This is not as obvious as it looks, because \( \hat{f}_{ab} \) is only defined as a limit in \( L^2 \).

**Exercise 6:** Check the formula for \( \hat{f}_{ab} \). (Hint: even though \( \hat{f}_{ab} \) is defined only as a limit in \( L^2 \), you can prove that the convergence also takes place pointwise and then use an earlier exercise about the equality of \( L^2 \) and pointwise limits.

**Exercise 7:** Check that \( \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \) for functions in \( L^2(\mathbb{R}) \) – this is Parseval’s identity (you did this before on \( S^1 \)).

**Exercise 8:** Check that \( \hat{fg} = \hat{f} \ast \hat{g} \) for functions in \( L^2(\mathbb{R}) \) (Use Exercise 7).

**AN EXAMPLE**

Next, we compute an interesting Fourier transform and use its inverse to calculate a couple of difficult integrals. Let \( f(x) \) be the indicator function of the interval \([-\frac{1}{2}, \frac{1}{2}]\), so \( f(x) = 1 \) for \(-\frac{1}{2} \leq x \leq \frac{1}{2} \) and \( f(x) = 0 \) otherwise. The Fourier transform of \( f \) is

\[ \hat{f}(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi x} dx = \frac{\sin \pi \xi}{\pi \xi}. \]

Because the \( f \in L^2(\mathbb{R}) \), we know that the inverse transfor of \( \hat{f} \) is \( f \) itself, in other words:

\[ f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i \xi x} d\xi = \int_{-\infty}^{\infty} \frac{\sin \pi \xi}{\pi \xi} e^{2\pi i \xi x} d\xi. \]

We can put \( x = 0 \) in this, since \( f(0) = 0 \) we have

\[ \int_{-\infty}^{\infty} \frac{\sin \pi \xi}{\pi \xi} d\xi = 1. \]

Next, we can use Plancherel’s formula to calculate

\[ 1 = \| f \|_2^2 = \int_{-\infty}^{\infty} \left( \frac{\sin \pi \xi}{\pi x} \right)^2 d\xi. \]
EIGENVALUES OF THE FOURIER TRANSFORM

We turn now to a look at the Fourier transform as a linear operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. In particular, since the Fourier transform is norm-preserving, it must be the case that all its eigenvalues have (complex) absolute value equal to 1. But we can do better than this, by noticing that, if you take the Fourier transform of the Fourier transform of a function $f(x)$, you get $f(-x)$ (because of the minus sign in the formula for the inverse transform). But this means that the fourth power of the Fourier transform is the identity map — this means that the fourth power of any eigenvalue of the Fourier transform is 1. So the only possible eigenvalues of the Fourier transform are $\pm 1$ and $\pm i$.

So can we find the eigenfunctions that go with these eigenvalues? We start with an important example: Let $f(x) = e^{-\pi x^2}$. We claim that $\hat{f}(\xi) = f(\xi)$. To show this, we can use the standard polar coordinate trick to show that

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = 1.$$  

Next, we’ll use the two “operational properties” of the Fourier transform:

$$\hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi)$$  

and

$$\hat{f}'(\xi) = -2\pi i x f(\xi)$$

as follows. The function $f(x) = e^{-\pi x^2}$ satisfies the ordinary differential equation $f' = -2\pi x f$, with the initial condition $f(0) = 1$. We’ve already shown that $\hat{f}(0) = 1$, and now we can use the operational property to conclude that

$$\hat{f}'(\xi) = -2\pi i x f(\xi) = i\hat{f}'(\xi) = -2\pi \xi \hat{f}(\xi),$$

so $\hat{f}$ satisfies the same initial-value problem as $f$. Thus the two functions must be the same.

So we have an eigenfunction for the eigenvalue 1, namely $f(x) = e^{-\pi x^2}$. The other eigenfunctions of the Fourier transform are related to this function — in fact, they are products of polynomials with these function. This was originally discovered by Charles Hermite, and so the functions are called Hermite functions and the corresponding polynomials are called Hermite polynomials. They all turn out to be given by the formula

$$h_n(x) = \frac{(-1)^n}{n!} e^{\pi x^2} \frac{d^n}{dx^n} e^{-2\pi x^2} \quad n = 0, 1, \ldots$$

and it is the case that

$$\hat{h}_n(\xi) = (-i)^n \hat{h}_n(\xi).$$
If we define $e_n$ to be the scaled Hermite functions

$$e_n(x) = \frac{1}{\|h_n\|_2} h_n(x) = \sqrt{\frac{\sqrt{2} n!}{4\pi}} h_n(x),$$

then the sequence $\{e_n\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. This gives us the alternative formula for the Fourier transform:

$$\hat{f}(\xi) = \sum_{n=0}^{\infty} \langle f, e_n \rangle (-i)^n e_n(\xi).$$

Therefore we may split $L^2(\mathbb{R})$ into the orthogonal direct sum of the eigenspaces corresponding to the four eigenvalues $\lambda_0 = 1$, $\lambda_1 = -i$, $\lambda_2 = -1$ and $\lambda_3 = i$:

$$L^2(\mathbb{R}) = H_0 \oplus H_1 \oplus H_2 \oplus H_3,$$

where

$$H_k = \{ f \in L^2(\mathbb{R}) \mid f = \sum_{n=0}^{\infty} \langle f, e_{4n+k} \rangle e_{4n+k} \}, \quad k = 0, 1, 2, 3,$$

in the sense that any $f \in L^2(\mathbb{R})$ can be expressed in precisely one way as a sum $f = f_0 + f_1 + f_2 + f_3$ of pieces, one from each eigenspace. You get to prove all this, in the following exercises:

**Exercise 9:** Show (by induction) that $h_n$ is the product of $e^{-\pi x^2}$ and a polynomial of degree $n$ for all $n = 0, 1, \ldots$. Explain why this implies that $h_n \in S(\mathbb{R})$.

**Exercise 10:** Prove that $h'_n - 2\pi x h_n = -(n+1)h_{n+1}$ (just use the definition of $h_n$ given above.

**Exercise 11:** If you define $h_{-1}$ to be the zero function, show that $h'_n + 2\pi x h_n = 4\pi h_{n-1}$. (It will help to prove and use the fact that for any sufficiently differentiable function $f$,

$$4\pi x \frac{d^n f}{dx^n} - \frac{d^n}{dx^n}(4\pi x f) = -4\pi n \frac{d^{n-1} f}{dx^{n-1}}.$$

**Exercise 12:** Show that $\widehat{h}_n = (-i)^n h_n$ and $\check{h}_n = i^n h_n$. (Use the operational properties to show that $\widehat{h}_n$ and $(-i)^n h_n$ satisfy the same recursion formula. That and the fact that $\widehat{h}_0 = h_0$ will finish the proof.

**Exercise 13:** Show that $h_n$ is also an eigenfunction of the differential operator $L$, defined by

$$L(f) = f'' - 4\pi^2 x^2 f$$

with eigenvalue $-4\pi (n + \frac{1}{2})$ (use problems 10 and 11).
Exercise 14: Show that $\langle h_n, h_m \rangle = 0$ whenever $n \neq m$. (Prove and use the fact that $L$ is a self-adjoint differential operator, $\langle Lh_n, h_m \rangle = \langle h_n, Lh_m \rangle$, but if $n = m$, these will be different).

Exercise 15: Show that the family $\{h_n\}$ spans $L^2(\mathbb{R})$. To do this, show that for any $f \in L^2(\mathbb{R})$, the Fourier transform of $G_f(x) = e^{-\pi x^2}f(x)$ can be expanded into a power series:

$$\hat{G}_f(\xi) = \int f(x)e^{-\pi x^2}e^{-2\pi i x \xi}dx = \int f(x)e^{-\pi x^2} \sum_{n=0}^{\infty} (-2\pi i \xi)^n \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-2\pi i \xi)^n}{n!} \int f(x)e^{-\pi x^2}x^n dx.$$  

Check that this is legitimate, and infer that $f \equiv 0$ is the only function that is perpendicular to all the Hermite functions ($f$ is perpendicular to all the Hermite functions if and only if all the integrals in the last expression are zero, if and only if the Fourier transform of $G_f$ is zero etc.)

Exercise 16: Use Exercise 13 and some integration by parts to show that

$$\int x^2|f(x)|^2dx + \int \xi^2|\hat{f}(\xi)|^2d\xi = \frac{1}{\pi} \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) |\langle f, e_n \rangle|^2 \geq \frac{1}{2\pi} \|f\|_2^2$$

and that the lower bound is achieved only for constant multiplies of $e^{-\pi x^2}$ (shades of Wirtinger!). The content of this inequality is that not both $f$ and $\hat{f}$ can be too “concentrated” near the origin. This is a precursor to the famous Heisenberg inequality:

$$\int x^2|f(x)|^2dx \times \int \xi^2|\hat{f}(\xi)|^2d\xi \geq \frac{1}{16\pi^2} \|f\|_4^4,$$

which we will prove later. The extra sharpness in Heisenberg comes from the fact that the arithmetic-geometric mean inequality. A hint for the current exercise is that for $f \in S(\mathbb{R})$,

$$4\pi \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) |\langle f, e_n \rangle|^2 = -\langle Lf, f \rangle$$

$$= \int |f'(x)|^2dx + 4\pi^2 \int x^2|f(x)|^2dx = 4\pi^2 \left( \int x^2|f(x)|^2dx + \int \xi^2|\hat{f}(\xi)|^2d\xi \right),$$

using integration by parts and the operational property.

Exercise 17: The last issue left is the normalization constant for the Hermite func-
tions. In other words, prove that

$$\|h_n\|_2^2 = \frac{(4\pi)^n}{\sqrt{2^n n!}}.$$  

To do this inductively, prove (using the recurrences for Hermite functions found in problems 10 and 11) and then use the fact that

$$(n + 1)\|h_{n+1}\|_2^2 = 4\pi\|h_n\|_2^2.$$