1. Let \( f(x) \) be a differentiable function that satisfies

\[
f(x + y) = f(x)f(y)
\]

for all real \( x \) and \( y \). Prove that \( f(x) = c^x \) where \( c = f(1) \). (This is essentially problem 6(a) on page 197 of Rudin.)

2. (An easy one if you approach it the right way.) Let \( A \) be a rectangle (box) in \( \mathbb{R}^n \), and let \( f \) and \( g \) be bounded, integrable functions on \( A \). Show that the product \( fg \) is also integrable.

3. A function \( f: \mathbb{R}^n \to \mathbb{R} \) is called **homogeneous of degree \( m \)** if for \( t \in \mathbb{R} \) with \( t \geq 0 \) and \( x \in \mathbb{R}^n \), we have

\[
f(tx) = t^m f(x).
\]

(a) Give examples of non-zero functions that are homogenous of degree \( m \) for \( m = 1, 2, 3 \).

(b) Show that if \( f \) is differentiable, as well as being homogeneous of degree \( m \), then

\[
\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = mf(x)
\]

for all \( x \in \mathbb{R}^n \).

4. Let \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) be differentiable functions for which \( f(1) = g(1) = 0 \). For what additional conditions on \( f \) and \( g \) will the implicit function theorem guarantee a solution of the equations

\[
f(xy) + g(yz) = 0, \quad g(xy) + f(yz) = 0
\]

for \( y \) and \( z \) as functions of \( x \) in a neighborhood of the point \((1, 1, 1)\)?

5. Are the functions \( f(x, y) = (x + y)/x \) and \( g(x, y) = (x + y)/y \) functionally dependent? Use the implicit function theorem to decide. If they are, then find a nontrivial function \( F \) so that \( F(f, g) = 0 \).
6. Consider the integral
\[ I(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx \]
as a function of the parameter \( a \), for \( a > 0 \).

(a) Explain why (i.e., prove that – and cite carefully any theorem you use) 
\[ \lim_{a \to \infty} I(a) = 0. \]

(b) Explain why \( I'(a) \) can be computed by differentiation under the integral sign.

(c) Use part (b) to calculate \( I'(a) \) — sorry about the integration by parts!

(d) Now use (a) and (c) to calculate \( I(a) \) for all \( a > 0 \).

(e) Extra credit: Justify the interchange of the (improper!) integral with the process of taking the limit as \( a \to 0 \) and hence evaluate 
\[ \int_0^\infty \frac{\sin x}{x} \, dx. \]

If you took Math 241 or 410, you may know a complex variables way to do this integral as well.

7. Consider the equation
\[ f(x, \varepsilon) = x^3 - 3x^2 + \varepsilon = 0. \]

(a) Show that when \( \varepsilon = 0 \), the equation \( f(x, 0) = 0 \) has two roots, and find them.

(b) For \( \varepsilon \) near zero, near which of the two roots from part (a) is the existence of a solution of \( f(x, \varepsilon) = 0 \) guaranteed by the implicit function theorem?

(c) Calculate \( \frac{dx}{d\varepsilon} \) at \( \varepsilon = 0 \) and \( x = \) the root from part (b).

(d) The implicit function theorem only tells you that the solution \( x(\varepsilon) \) exists only for \( \varepsilon \) near zero. Now, let’s show that the function \( x(\varepsilon) \) can be extended continuously (even smoothly) all the way to \( \varepsilon = 1 \). Do this as follows:

(i) Prove the following using the implicit function theorem: If \( (x_0, \varepsilon_0) \) is a point in \( \mathbb{R}^2 \) for which \( f(x_0, \varepsilon_0) = 0 \), \( 0 \leq \varepsilon \leq 1 \), and \( 2.5 \leq x_0 \leq 4 \), then for all \( \varepsilon \) near \( \varepsilon_0 \), there is an \( x \) near \( x_0 \) such that \( f(x, \varepsilon) = 0 \), and in fact \( x \) is a continuous (smooth) function of \( \varepsilon \) near \( \varepsilon_0 \).

(ii) Let \( I \) be the subinterval of the \( \varepsilon \) interval \([0, 1]\) on which the solution \( x(\varepsilon) \) from part (b) exists as a continuous function. Explain why part (d,i) implies that \( I \) is open.
(iii) Using any method (think “Math 104”), prove that for all $\varepsilon$ between 0 and 1, there is exactly one solution $x$ of $f(x, \varepsilon) = 0$ such that $2.5 \leq x \leq 4$.

(iv) Now suppose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is a convergent (to $\varepsilon^*$) sequence of values of $\varepsilon$ between 0 and 1 such that for each $\varepsilon_i$ there is an $x_i$ between 2.5 and 4 such that $f(\varepsilon_i, x_i) = 0$. Explain why there must be at least a subsequence of the $x_i$ that converges to a number $x^*$ that satisfies $2.5 \leq x^* \leq 4$.

(v) Explain why it must be the case that $f(x^*, \varepsilon^*) = 0$.

(vi) Explain why parts (d, iv) and (d, v) imply that the interval $I$ from part (d, iii) is closed.

(vii) Explain why parts (d, iii) and (d, vi) together imply that the interval $I$ is all of $[0, 1]$. Therefore there is a continuous path $(x(\varepsilon), \varepsilon)$ from the solution you found in part (b) for $\varepsilon = 0$ to the (unknown, but existent) solution for $\varepsilon = 1$.

This is called the continuity method for proving existence of solutions to nonlinear equations, and is often used in the study of ordinary and partial differential equations.