Solutions to Select Problems of HW #2

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Section 2.3 Problem 4

Problem

If G is a group in which $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive integers i for all $a, b \in G$, show that G must be abelian

Solution

Let $a, b \in G$, and $i \in \mathbb{Z}$ be such that $(ab)^i = a^i b^i$, $(ab)^{i+1} = a^{i+1}b^{i+1}$ and $(ab)^{i+2} = a^{i+2}b^{i+2}$. We show that this implies that $ab = ba$.

First observe that

$$a^{i+1}b^{i+1} = (ab)^{i+1} = (ab)^i(ab) = a^i b^i ab$$

multiply on the left by a^{-i} and on the right by b^{-1}, to get

$$ab^i = b^i a.$$

Similarly, we have

$$a^{i+2}b^{i+2} = (ab)^{i+2} = (ab)^{i+1}(ab) = a^{i+1}b^{i+1}ab$$

so

$$ab^{i+1} = b^{i+1}a.$$

We rewrite this as

$$(ab)b^i = ab^{i+1} = b^{i+1}a = b(b^i a) = b(ab^i) = (ba)b^i,$$

multiplying by b^{-i} on the right we get $ab = ba$.

Since a and b were arbitrary element of G, we conclude that every pair of elements in G commutes, so G is abelian.

Section 2.5 Problem 3

Problem

If G has no nontrivial subgroups, show that G must be finite of prime order.

Solution

Note: this problem assumes G is not the trivial group $G = (e)$.

Take some $x \in G$, $x \neq e$, then $(x) \subset G$ is a subgroup of G, and $(x) \neq (e)$, so $(x) = G$. Therefore, G is a cyclic group. We now have two cases, either $x^2 = e$, and $(x) = G$ has only 2 elements (which completes the proof), or $x^2 \neq e$. If $x^2 \neq e$, then consider the subgroup $(x^2) \subset G$, we once again have $(x^2) \neq (e)$, so we must have $(x^2) = G$. Therefore, in particular, $x \in (x^2)$, so $x = (x^2)^k$ for some $k \in \mathbb{Z}$. Therefore, $x = x^{2k}$ and so $x^{2k-1} = e$. So, $o(x) \mid 2k - 1$, and therefore, $(x) = G$ is finite.

Next, suppose that $o(G) = n = a \cdot b$ with $a > 1, b > 1$ integers. We know that $G = (x)$ for some $x \in G$ $x \neq e$, so $o(x) = n$. Consider the subgroup $(x^a) \subset G$, observe that $(x^a)^b = x^{ab} = x^n = e$, so $o(x^a) < n$. Therefore, $(x^a) \neq G$ and $(x^a) \neq (e)$, which contradicts our assumption that G has no nontrivial subgroups. Therefore, we must have n prime.

Section 2.5 Problem 21

Problem

Let the mapping τ_{ab} for a, b real numbers, map the reals into the reals by the rule $\tau_{ab} : x \to ax + b$. Let $G = \{\tau_{ab} | a \neq 0\}$. Prove that G is a group under the composition of mappings. Find the formula for $\tau_{ab}\tau_{cd}$.

Solution

We start with the composition formula:

$$\tau_{ab}\tau_{cd}(x) = \tau_{ab}(cx + d) = a(cx + d) + b = acx + ad + b = \tau_{ac,ad+b}(x)$$

We verify associativity:

$$\begin{aligned}\tau_{ab}(\tau_{cd}\tau_{ef}) &= \tau_{ab}\tau_{ce,cf+d} = \tau_{ace,acf+ad+b} \\ &= \tau_{ac,ad+b}\tau_{ef} = (\tau_{ab}\tau_{cd})\tau_{ef}.\end{aligned}$$

We claim that $\tau_{1,0}$ is the idenity, we verify this:

$$\begin{aligned}\tau_{1,0}\tau_{ab} &= \tau_{1 \cdot a, 1 \cdot b + 0} = \tau_{ab} \\ \tau_{ab}\tau_{1,0} &= \tau_{a \cdot 1, a \cdot 0 + b} = \tau_{ab}.\end{aligned}$$

We next verify that $\tau_{ab}^{-1} = \tau_{1/a,-b/a}$:

$$\begin{aligned}\tau_{ab}\tau_{1/a,-b/a} &= \tau_{a(1/a),a(-b/a)+b} = \tau_{1,0} \\ \tau_{1/a,-b/a}\tau_{ab} &= \tau_{(1/a)a,(1/a)b-b/a} = \tau_{1,0}.\end{aligned}$$

Therefore, G is a group with the composition defined as above.