\section*{\textbf{Q-FACTORIAL LAURENT RINGS}}

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\textbf{Abstract.} Dolgachev proves that the ring naturally associated to a generic Laurent polynomial in $d$ variables, $d \geq 4$, is factorial \cite{Dolgachev} (for any field $k$). We prove a sufficient condition for the ring associated to a very general complex Laurent polynomial in $d=3$ variables to be $\mathbb{Q}$-factorial.

\section{1. Introduction}

In \cite{Dolgachev} Dolgachev proves that the ring $A_F$ naturally associated to generic Laurent polynomial $F$ in $d$ variables, $d \geq 4$, with coefficients in any field $k$, is factorial. The basic ingredient in Dolgachev’s proof is Grothendieck’s Lefschetz-type theorem (\cite{Grothendieck}, Prop. 3.12) which, among other things, shows that under suitable conditions, the natural restriction map $\text{Pic}(X) \to \text{Pic}(Y)$, where $X$ is a scheme and $Y$ is subvariety corresponding to an ideal sheaf in $\mathcal{O}_X$, is an isomorphism. This result can be applied only when $d \geq 4$.

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In this paper we consider the case \( d = 3 \), assuming that \( k = \mathbb{C} \), and prove a sufficient condition for the ring \( A_F \) to be \( \mathbb{Q} \)-factorial (Theorem 3.1). The proof of this fact follows the lines of Dolgachev’s proof, with Grothendieck’s result replaced by a Noether-Lefschetz theorem for hypersurfaces in toric 3-folds (Theorem 2.5) that we proved in [2].

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### 2. Preliminaries

We follow the notation in [1] and [2]. Let \( M \) be a \( d \)-dimensional lattice, \( N = \text{Hom}(M, \mathbb{Z}) \) and \( T(\Sigma) = N \otimes \mathbb{C}^* \) the associated algebraic torus. Let \( \Sigma \subset N^* \) be a complete simplicial fan, and denote by \( \mathbb{P}_\Sigma \) the corresponding complete toric variety. The torus \( T(\Sigma) \) naturally acts on \( \mathbb{P}_\Sigma \); \( T_\tau \subset \mathbb{P}_\Sigma \) denotes the orbit of a subset of \( \mathbb{P}_\Sigma \) corresponding to a face \( \tau \) of \( \Sigma \) under this action; the open dense orbit is denoted by \( \mathbb{P}_0^\Sigma \).

**Definition 2.1.** [1, Def. 4.13] A hypersurface \( X \) in \( \mathbb{P}_\Sigma \) is nondegenerate if \( X \cap T_\tau \) is a smooth 1-codimensional subvariety of \( T_\tau \) for all faces \( \tau \) in \( \Sigma \).

\( \mathbb{P}_\Sigma \) has only abelian quotient singularities, and is therefore an orbifold.

**Proposition 2.2.** [1, Prop. 3.5, 4.15] Let \( L \) be an ample line bundle on \( \mathbb{P}_\Sigma \). The hypersurface \( X \subset \mathbb{P}_\Sigma \) given by the zero locus of a generic section of \( L \) is nondegenerate. Moreover, \( X \) is an orbifold.

Since \( X \) is an orbifold, its complex cohomology has a pure Hodge structure [8]. This is an essential point in the proof of our Theorem 2.5.

**Definition 2.3 (The Cox Ring [3]).** Consider a variable \( z_i \) for each 1-dimensional cone \( \zeta_i, i = 1, \ldots, n \) in \( \Sigma \), and let \( S(\Sigma) = \mathbb{C}[z_1, \ldots, z_n] \).

The Cox ring has a natural gradation given by its class group \( Cl(\Sigma) \) of \( \mathbb{P}_\Sigma \).

Let \( L \) be an ample line bundle on \( \mathbb{P}_\Sigma \), and let \( f \in H^0(\mathbb{P}_\Sigma, L) \simeq S(\Sigma)_\beta \), where \( \beta = \text{deg}(L) \).

**Definition 2.4.** The Jacobian ring of \( f \) is the quotient \( R(f) = S(\Sigma)/J(f) \), where \( J(f) \) is the ideal in \( S(\Sigma) \) generated by the derivatives of \( f \).
The Jacobian ring $R(f)$ inherits a natural gradation from $S(\Sigma)$.

The next theorem was proved in [2], and will be key to proving our result about Laurent rings. We assume $d = 3$. We recall that the Picard number is defined as the rank of the class group.

**Theorem 2.5.** [2] Let $P_{\Sigma}$ a complete simplicial toric variety, and $X \subset P_{\Sigma}$ a very general hypersurface cut by a section $f$ of an ample line bundle $L$ such that the multiplication morphism

$$R(f)_{\beta} \otimes R(f)_{\beta_0} \rightarrow R(f)_{2\beta - \beta_0}$$

is surjective (here $\beta = \text{deg}(L)$ and $\beta_0 = -\text{deg}(K_{P_{\Sigma}})$, where $K_{P_{\Sigma}}$ is the canonical bundle of $P_{\Sigma}$). Then $X$ has the same Picard number as $P_{\Sigma}$.

Recall that a property is very general if it holds in the complement of countably many proper subvarieties.

### 3. Q-factorial Laurent rings

The ring $\mathbb{C}[M]$ may be identified with the ring of regular functions on the torus $T(\Sigma) \simeq \mathbb{P}^0_{\Sigma} \subset P_{\Sigma}$. An element $F \in \mathbb{C}[M]$ is called a Laurent polynomial; $F$ may be regarded as a section of $L$, and it defines a hypersurface $X_F$ in $P_{\Sigma}$.

Let $\Delta \subset M \otimes \mathbb{R}$ be the polytope uniquely determined by the fan $\Sigma$ (see [7], Lemma 2.3). To each Laurent polynomial $F$ on can associate a polytope $\Delta_F$, called the Newton polytope of $F$. This is most easily described by choosing an isomorphism $M \simeq \mathbb{Z}^d$, writing

$$F = \sum_{i_1, \ldots, i_d \in \mathbb{Z}^d} a_{i_1, \ldots, i_d} t_1^{i_1} \cdots t_d^{i_d}$$

and defining

$$\text{supp}(F) = \{i_1, \ldots, i_d \in \mathbb{Z}^d | a_{i_1, \ldots, i_d} \neq 0\}.$$ 

$\Delta_F$ is then defined to be the convex hull of $\text{supp}(F)$ and $\Gamma(\Delta)$ the set of all Laurent polynomials such that $\Delta_F \subset \Delta$. $\Gamma(\Delta)$ is a finite dimensional vector space over $\mathbb{C}$.

By results given in [6] (see also [7], Chapter 2) a Laurent polynomial $F$ extends to a meromorphic function on $P_{\Sigma}$, which is a section of an ample line bundle $L_F$. Thus, $F$ may be regarded as an element in $S(\Sigma)_\beta$, where $\beta = \text{deg}(L_F)$. Denote by $A_F$ the ring $\mathbb{C}[M]/(F)$.
Theorem 3.1. Let \( d = 3 \), and let \( F \) be a very general Laurent polynomial in \( \Gamma(\Delta) \); set \( \beta = \deg(L_F) \) and \( \beta_0 = -\deg(K_{P_F}) \). If the multiplication morphism

\[
R(F)_{\beta} \otimes R(F)_{\beta-\beta_0} \rightarrow R(F)_{2\beta-\beta_0}
\]

is surjective, the ring \( A_F \) is \( \mathbb{Q} \)-factorial.

The proof that \( A_F \) is \( \mathbb{Q} \)-factorial follows closely the proof of Theorem 1.1 in [4]. The basic idea is to formulate the problem in a geometric way:

**Proof.** Let \( X_F \subset P_\Sigma \) be the hypersurface cut by \( F \) (as a section of \( L_F \)). By Proposition 2.2 the hypersurface \( X_F \) is nondegenerate, and is an orbifold.

Note that the ring \( A_F \) may be identified with the ring of regular functions on the affine part \( U_F = X_F \cap P^0_\Sigma \) of \( X_F \). Since the Picard group of \( P^0_\Sigma \) is trivial, every Cartier divisor in \( P_\Sigma \) is linearly equivalent to a divisor supported in \( P_\Sigma - P^0_\Sigma \). By Theorem 2.5, \( X_F \) has the same Picard number as \( P_\Sigma \), i.e., \( \rho(X_F) = \rho(P_\Sigma) \). Then any Cartier divisor in \( X_F \) is linearly equivalent modulo torsion to a divisor supported in \( X_F - U_F \), so that \( \text{Pic}(U_F) \otimes \mathbb{Q} = 0 \). Since \( U_F \) is normal (actually smooth), then \( \text{Cl}(U_F) \otimes \mathbb{Q} = 0 \). As \( U_F \simeq \text{Spec}(A_F) \), we have \( \text{Cl}(A_F) \otimes \mathbb{Q} = 0 \).

\[ \square \]

References


