Sublevel Set Estimates in Higher Dimensions: Symmetry and Uniformity

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One-Dimensional Sublevel Set Theory

A Successful One-Dimensional Program

- For a very long time now, there has been a recognized pathway to estimate one-dimensional scalar oscillatory integrals ("First Kind" in Stein's classification), e.g.,

$$I(\lambda) := \int e^{i\lambda\phi(t)} a(t) dt$$

  The process combines van der Corput's lemma for non-stationary phase intervals and sublevel set estimates near critical points of the phase.
- The method of TT^* allows these ideas to extend to oscillatory integral operators in many cases.

Sample Theorem

Suppose f is C^2 on $I := (a, b)$ s.t. I is a union of $\leq n$ nonoverlapping intervals on which f'' is either nonpositive or nonnegative. Then there is $E \subset I$ which is a union of $\leq n$ disjoint intervals s.t. $|E| \geq c_n|I|$ and

$$|I| \sup_{t \in E} \left| \frac{df}{dt}(t) \right| \leq C_n \sup_{t \in I} |f(t)|.$$

Proof: Take the decomposition of I and clip ears of length $|I|/(4n)$ off the ends of each subinterval. Those which have any remaining length are the intervals of E. Then use the FTC and monotonicity of f' to bound $\sup f'$ on each remaining interval by $\sup f$ on the ears.

Corollary

If $f \in C^{k+1}(I)$ and $f^{(k)} \geq 1$ on I, then for $\epsilon > 0$,

$$|\{t \in I \ : \ |f(t)| \leq \epsilon\}| \leq C_k \epsilon^{1/k}.$$

- In the words of Carbery, "'If a real-valued function u has a large derivative, then it cannot spend too much time near any one value.'"
- The frustrating thing is that in 2D and above, there are few simple and elegant results which capture this idea in a way that's even remotely sharp.
  - Phong-Stein-Sturm Example $(x_1 + \cdots + x_d)^k$: Knowing $\partial^\alpha f(x) \geq 1$ on $[0,1]^d$ only implies sublevel set estimates $\lesssim \epsilon^{1/|\alpha|}$, which is approximately what you get by using the 1D result and Fubini's Theorem.

- The Phong-Stein-Sturm example unfortunately shows that there's no added benefit to knowing that several mixed partial derivatives have lower bounds.
- Likewise, lower bounds for linear partial differential operators give estimates don't give anything extra over individual monomials.

I claim that the discrepancy is due to the fact we can only really prove interesting results in 1D for intervals, but all sublevel sets in 1D are close enough to being intervals that nobody notices the difference.

"Close Enough" = contain an interval of length comparable to the measure of the set.

Higher Dimensions: Carbery's Theorem

Theorem (Carbery, Contemp. Math. 2010)

Suppose that f is a nonnegative, strictly convex C^2 function on a convex body $K \subset \mathbb{R}^n$. If $\det \nabla^2 f \geq 1$, then

$$\left|\{x \in K \ : \ u(x) \leq \epsilon\}\right| \leq C_n \epsilon^{n/2}.$$

A Proof: **Step 1:** By the John Ellipsoid Theorem and an affine transformation, we may assume that there is a Euclidean ball B contained in the sublevel set K_ϵ such that $|B| \geq c_n |K_\epsilon|$.

Step 2: Like the 1D case, integrate along lines

$$|B|^{1/n} \sup_{x \in \frac{1}{2}B} |\nabla f(x)| \leq C_n \sup_{x \in B} |f(x)| \leq C_n \epsilon.$$

Step 3: Strict convexity implies that $x \mapsto \nabla f(x)$ is injective on B.

Step 4: Change of variables formula:

$$\int_{\frac{1}{2}B} |\det \nabla^2 f(x)| dx = \left| \nabla f \left(\frac{1}{2} B \right) \right|$$

Step 5: Derivative estimate bounds volume:

$$\left| \nabla f \left(\frac{1}{2} B \right) \right| \leq (C_n |B|^{-1/n} \epsilon)^n.$$

Step 6: Conclusion:

$$c_n |B| \leq \int_{\frac{1}{2}B} |\det \nabla^2 f(x)| dx \leq (C_n |B|^{-1/n} \epsilon)^n.$$

Derivative Estimates Express Geometry

- To simplify matters somewhat, I will focus attention specifically on polynomial functions f on $\mathbb{R}^n$ of deg $\leq d$. We can upgrade the 1D result a little bit:

$$|I|^s \sup_{t \in I} |p^{(s)}(t)| \leq C_{d,s} \sup_{t \in I} |p(t)|.$$

Corollary

Suppose $|\frac{\partial^{s_1}}{\partial x_1^{s_1}} f(x)|^{1/s_1} \cdots |\frac{\partial^{s_n}}{\partial x_n^{s_n}} f(x)|^{1/s_n} \geq 1$ on open set $U \subset \mathbb{R}^n$. Then the sublevel set $\{x \in U \ : \ |f(x)| \leq \epsilon\}$ contains no axis parallel box of volume $\geq C_{n,k,s} \epsilon^{1/s_1 + \cdots + 1/s_n}$.

A Geometric Framework for Derivative Estimates

- Nonlinear quantities of this sort when expressed in terms of partial derivatives $\frac{\partial}{\partial x_i}$ actually give **sharp and uniform information about the largest axis-parallel boxes which fit inside a sublevel set.**
- If the differential operator has coordinate symmetry, the quantity measures the size of boxes of arbitrary orientation which fit inside the sublevel set.
- In the case of convexity, sublevel sets are always approximately boxes.
- To push beyond “box content,” one needs differential quantities with stronger symmetry closer to diffeomorphism invariance.

Moving Further into Higher Dimensions

There are three basic questions I would like to consider:

- Can one understand what some “right” nonlinear differential quantities are?
- Can one successfully extend the differential inequality/geometric correspondence?
- Can one get uniform integral estimates in addition to sublevel set estimates?

In each case, the answer is a qualified “yes.”

Preparation: Nash Functions

A **Nash function** on an open set $U \subset \mathbb{R}^n$ is an analytic function f which satisfies a nontrivial polynomial equation $p(x, f(x)) = 0$. Things to know:

- There is a notion of complexity (c.f. degree) and a Bézout Theorem for bounding the number of nondegenerate systems of Nash equations.
- Basic algebraic operations preserve Nash, as do coordinate partial derivs. (with bdd. complexity).
- Equivalent definition: f is a polynomial in x and Φ for some analytic $\Phi : U \to \mathbb{R}^M$ s.t. $p(x, \Phi(x)) = 0$ for some poly. $p(x, y)$ with $(\det \frac{\partial p}{\partial y})(x, \Phi(x)) \neq 0$ on any open set.

General Derivative Estimates

Lemma

Let $U_0 \subset \mathbb{R}^d$ be open and suppose that $f : U_0 \to \mathbb{R}^m$ is Nash and has Jacobian $D_x f$ which is everywhere rank d. Let $E_0 \subset U_0$ be compact s.t. $\sup_{x \in E_0} |f^j(x)| \leq 1$ for all $j = 1, \ldots, m$ and let w be a nonnegative weight on U_0 For each integer $N \geq 1$, there exists an open set $U_N \subset U_{N-1}$, a compact set $E_N \subset E_{N-1} \cap U_N$, smooth vector fields $\{X_i^{(N)}\}_{i=1}^d$ defined on U_N, and a positive constant $c_{N,d}$ depending only on N and d such that the following are true:

(Lemma continued)

- For each $N \geq 1$, $w(E_N) \gtrsim w(E_0)$.
- For each N', N with $1 \leq N' < N$ and each $x \in U_N$, the $\{X_i^{(N)}\}$ are smooth linear combinations of the $\{X_{i'}^{(N')}\}$ with for some coefficients $c_i^{i'}$ of magnitude ≤ 2.
- For each $N \geq 1$ and all for all $x \in E_N$,

$$w(x) \left| \det(X_1^{(N)}, \ldots, X_d^{(N)}) \right| \Big|_x \gtrsim w(E_0).$$

- For each $N \geq 1$, each generalized multiindex α of generation at most N, and each $j \in \{1, \ldots, m\}$,

$$\sup_{x \in U_N} |X^\alpha f^j(x)| \leq 1.$$

Proof Sketch

- Proof is by induction on N. At each stage, there is some open set U_N, some finite list of Nash functions $f^1, \ldots, f^M$ on U_N, and some compact set $E_N \subset E \cap U_N$ whose measure $w(E_N)$ is comparable to $w(E)$.
- **Step 1:** Subdivide U_N into finitely many pieces depending on which tuple $j_1, \ldots, j_d$ maximize

$$\det \frac{\partial(f^{j_1}, \ldots, f^{j_d})}{\partial x}.$$

Define vector field X_i by letting

$$X_i f := \det \frac{\partial(f, f^{j_1}, \ldots, \widehat{f^{j_i}}, \ldots f^{j_d})}{\partial x} \Big/ \det \frac{\partial(f^{j_1}, \ldots, f^{j_d})}{\partial x}.$$

- This definition makes the pointwise bound $|X_i f| \lesssim 1$ trivial.
- **Step 2:** The (only slightly) harder part is establishing that the vector fields X_i are not trivial (e.g., linearly dependent).
- We use Change of Variables:

$$\int_{E_N} \left| \det \frac{\partial (f^{j_1}, \ldots, f^{j_d})}{\partial x} \right| dx \lesssim 1.$$

Chebyshev-type estimate says that

$$w^{-1} \left| \det \frac{\partial (f^{j_1}, \ldots, f^{j_d})}{\partial x} \right|$$

isn't $\gtrsim 1/w(E_n)$ on a set E' with $w(E') \geq \frac{1}{2} w(E_n)$.

- Let E_{N+1} be the complement of the set E'; here

$$w^{-1}\left|\det \frac{\partial(f^{j_1},\ldots,f^{j_d})}{\partial x}\right| \lesssim \frac{1}{w(E)}.$$

But $X_i f^{j_k} = \delta_{jk}$, so

$$w(E) \lesssim w\left|\det(X_1,\ldots,X_d)\right|.$$

- We can get estimates on the change of basis matrix between our $X_1,\ldots,X_d$ and the vector fields from previous steps just because we know that there are functions $f^{k_1},\ldots,f^{k_d}$ in the list such that

$$X_i^{new} = \sum_{j=1}^{d}(X_i^{new} f^{k_j})X_j^{old}.$$

Upgraded Sublevel Set Estimates

Example 1: Sublevel Sets

Theorem

Suppose $f_1, \ldots, f_n$ are degree $\leq d$ polys on $\mathbb{R}^n$. Then

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \geq 1 \text{ on } E \subset \mathbb{R}^n \Rightarrow$$

$$\left|\left\{x \in E \ : \ (f_1(x))^2 + \cdots + (f_n(x))^2 \leq \epsilon^2\right\}\right| \leq C_{n,d}\epsilon^n.$$

Proof: Change of variables formula combined with Bézout's Theorem.

Example 2: Norm Integrability

Theorem

Suppose $f_1, \dots, f_{n+1}$ are degree $\leq d$ polys on $\mathbb{R}^n$. Then

$$Q(f) := \det \begin{bmatrix} f_1 & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & \frac{\partial f_{n+1}}{\partial x_1} & \cdots & \frac{\partial f_{n+1}}{\partial x_{n+1}} \end{bmatrix} \geq 1 \text{ on } E \subset \mathbb{R}^n \Rightarrow$$

$$\int_E \frac{dx}{[(f_1(x))^2 + \cdots + (f_{n+1}(x))^2]^{n/2}} \leq C_{n,d}.$$

Proof: 1. Use $Q(f\varphi) = \varphi^{n+1} Q(f)$ for scalar functions φ.

2. Apply previous result to $f/||f||$.

Affine Curvature “Plus”

Setup: An n-dimensional submanifold of $\mathbb{R}^N$; equivalently, N functions $f_1, \ldots, f_N$ on $\mathbb{R}^n$.

- $\alpha := (\alpha_0, \alpha_1, \ldots)$: any ordering of all multiindices on $\mathbb{R}^n$ with nondecreasing degree: $0 = |\alpha_0| \leq |\alpha_1| \leq \cdots$.
- $X_1, \ldots, X_n$: linear combos of $\partial_{x_1}, \ldots, \partial_{x_n}$ with $|\det(X_1, \ldots, X_n)| = 1$.
- “Affine curvature plus” is defined to equal

$$Q_N(f) := \inf_X \max_\alpha \left| \det \begin{bmatrix} X^{\alpha_0} f_1 & \cdots & X^{\alpha_N} f_1 \\ \vdots & \ddots & \vdots \\ X^{\alpha_0} f_N & \cdots & X^{\alpha_N} f_N \end{bmatrix} \right| .$$

Properties of Affine Curvature Plus

- $|\alpha_0| + \cdots + |\alpha_N| := D$ depends only on N and n.
- For any smooth vector fields $Z_1, \ldots, Z_n$,

$$\max_{\alpha} \left| \det \begin{bmatrix} Z^{\alpha_0} f_1 & \cdots & Z^{\alpha_N} f_1 \\ \vdots & \ddots & \vdots \\ Z^{\alpha_0} f_N & \cdots & Z^{\alpha_N} f_N \end{bmatrix} \right| \geq |\det(Z_1, \ldots, Z_n)|^{\frac{D}{n}} Q_N(f).$$

- If $A \in \mathrm{SL}(N, \mathbb{R})$, $Q_N(Af) = Q_N(f)$.
- (NEW!) For any smooth function φ, $Q_N(\varphi f) = \varphi^N Q_N(f)$.

Theorem (Sublevel Set Style)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^N$ is polynomial of degree $\leq d$. Let weight $w := (Q_N(f))^{n/D}$. If $K \subset \mathbb{R}^N$ is a convex body and $E := f^{-1}(K)$, then

$$w(E) \leq C_{N,n,d}|K|^{n/D}.$$

Theorem (Enhanced Inequality)

Under the same hypotheses, for any $A \in \mathrm{SL}(N, \mathbb{R})$,

$$\int \frac{(Q_N(f))^{n/D}dx}{||Af||^{Nn/D}} \leq C_{N,n,d}.$$

Example

For any polynomial $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ of bounded degree,

$$\int \frac{|\det(\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t))|^{\frac{2}{n(n+1)}} dt}{||A\gamma(t)||^{\frac{2}{n}}} \lesssim 1$$

- The proof of the integral inequality is almost a trivial consequence of the sublevel set inequality combined with the additional symmetry: apply sublevel set estimate for K equal to the unit ball and replace f by $f/||f||$.

What's Next

- These estimates apply to vector-valued f; They have direct application to the establishment of certain endpoint L^p-improving inequalities for averages over hypersurfaces. A more desirable case would be k-form-valued functions f.
- Exploring potential applications to oscillatory integrals of the first kind in higher dimensions.
- Also significant would be to understand the underlying incidence problem. Are local bounds implied by nonvanishing of the differential operator in the smooth category?

Thank you for your attention.