An Introduction to Matrix Monotonicity, Realization Theory, and Non-commutative Function Theory

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1-variable Matrix Monotonicity

Let $E \subseteq \mathbb{R}$ be an interval and let $f : E \to \mathbb{R}$. Let A be an $n \times n$ self-adjoint matrix with spectrum $\sigma(A) \subseteq E$. Then

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*, \text{ then } f(A) = U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} U^*.$$

f is n-**matrix monotone on** E if whenever A, B are $n \times n$ self-adjoint matrices with spectrum $\sigma(A), \sigma(B) \subseteq E$,

$$A \leq B \ (B - A \text{ is positive semidefinite}) \quad \text{implies} \quad f(A) \leq f(B).$$

Ex. $f(x) = c + dx$, where $d \in [0, \infty)$. Then if $A \leq B$,

$$f(B) - f(A) = (cI + dB) - (cI + dA) = d(B - A) \geq 0.$$

f is **matrix monotone on** E if f is n-matrix monotone for all n.

Loewner's Theorem

Loewner's Theorem- Part 1

Let $n \geq 2$. A function $f : E \to \mathbb{R}$ is n-matrix monotone on E if and only if f is differentiable on E and for every distinct list $\{\lambda_1, \ldots, \lambda_n\} \subseteq E$, the divided difference matrix

$$M_{ij} = \begin{cases} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} & \text{if } i \neq j \\ f'(\lambda_i) & \text{if } i = j \end{cases}$$

is positive semi-definite.

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. A **Pick function** is a holomorphic function mapping $\mathbb{H}$ into $\overline{\mathbb{H}}$.

Loewner's Theorem- Part 2

A function $f : E \to \mathbb{R}$ is matrix monotone on E if and only if f analytically continues to $\mathbb{H}$ as a map $f : \mathbb{H} \cup E \to \overline{\mathbb{H}}$ in the Pick class.

Ex 1. $f(x) = c + dx + \sum_{i=1}^{m} \frac{t_i}{\lambda_i - x}$ where $c \in \mathbb{R}$, $d, t_1, \ldots, t_m \geq 0$, and $\lambda_1, \ldots, \lambda_m \in \mathbb{R} \setminus E$.

Loewner's Theorem

Loewner's Theorem- Part 2

A function $f : E \to \mathbb{R}$ is matrix monotone on E if and only if f analytically continues to $\mathbb{H}$ as a map $f : \mathbb{H} \cup E \to \overline{\mathbb{H}}$ in the Pick class.

Examples: $\log x$, $\sqrt{x}$, $\tan x$. **Non-Examples:** $e^x, x^3, \sec x$

- Many known proofs (see Barry Simon's book *Loewner's Theorem on Monotone Matrix Functions*)
- A **key idea** is the use of **Nevanlinna representations** for Pick functions.

A function $f : \mathbb{H} \to \mathbb{C}$ is a Pick function if and only if there is $a \in \mathbb{R}$, $b \geq 0$ and a finite positive Borel measure μ on $\mathbb{R}$ such that

$$f(z) = a + bz + \int \frac{1 + tz}{t - z} d\mu(t).$$

The complement of the support of μ is exactly the set where f analytically continues to be real valued. This is a **Nevanlinna Representation** for f.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.The **Schur class** is the set of holomorphic $\phi : \mathbb{D} \to \overline{\mathbb{D}}$.

A Schur function ϕ is **inner** if $\lim_{r \nearrow 1} |\phi(r\tau)| = 1$ for a.e. $\tau \in \mathbb{T} = \partial\mathbb{D}$.

The **Cayley transform** $\alpha : \mathbb{D} \to \mathbb{H}$ is defined by $\alpha(z) = i\left(\frac{1+z}{1-z}\right)$.

- f is a Pick function iff $\phi = \alpha^{-1} \circ f \circ \alpha$ is Schur function.
- f is real-valued on $E \subseteq \mathbb{R}$ iff ϕ is unimodular on $\alpha^{-1}(E)$ and omits 1.

Realization Theory

Each Schur function $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ has a **transfer function realization**, i.e.

$$\phi(z) = A + B(1 - zD)^{-1}zC \quad \text{for } z \in \mathbb{D}, \quad \text{where}$$

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix}$$

is a contraction on a Hilbert space $\mathbb{C} \oplus \mathcal{M}$. The operator U can be chosen to be isometric, coisometric, or unitary.

With "minimal" $\mathcal{M}$ and U, this extends to any open $I \subseteq \mathbb{T}$ where ϕ is unimodular.

Two-variable Realization Theory

(Agler '90, Kummert '89): Each Schur function $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ possess a **transfer function realization**, i.e.

$$\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad \text{for } z \in \mathbb{D}^2, \quad \text{where}$$

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix}$$

is a contraction on a Hilbert space $\mathbb{C} \oplus \mathcal{M}$. The operator U can be chosen to be isometric, coisometric, or unitary. $\mathcal{M}$ decomposes as $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $E_z = z_1 P_1 + z_2 P_2$ where each P_j is the projection onto $\mathcal{M}_j$.

Applications on $\mathbb{D}^2$

- (Agler, '90) Nevanlinna-Pick Interpolation
- (Knese, '07) Infinitesimal Schwarz Lemma
- (Agler-McCarthy-Young, '12) Julia-Carathéodory Theorem
- (Agler-McCarthy-Young, '12) Loewner's theorem
- (B.-Pascoe-Sola, '18-19) Quantify boundary behavior of rational functions

Free or NC Sets

NC Function Theory: Began with J. Taylor and has had a meteoric recent resurgence.

Let W^d denote the d-dimensional matrix universe $W^d := \cup_{n=1}^{\infty} M_n(\mathbb{C})^d$.

A set $D \subseteq W^d$ is a **free set** if

- D is closed with respect to direct sums.
  $X = (X_1, \ldots, X_d), Y = (Y_1, \ldots, Y_d) \in D$ if and only if

$$\begin{bmatrix} X & \\ & Y \end{bmatrix} = \left(\begin{bmatrix} X_1 & \\ & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & \\ & Y_d \end{bmatrix} \right) \in D.$$

- D is closed with respect to unitary similarity. If $X \in D \cap M_{n\times n}(\mathbb{C})^d$ and U is an $n \times n$ unitary, then $UXU^* = (UX_1U^*, \ldots, UX_dU^*) \in D$.

Ex. The set of tuples $(X_1, \ldots, X_d)$ of positive semi-definite matrices (of the same size) is a free set in W^d.

Free or NC Functions

Let D be a free set.

Let $f : D \to W^1$ is a **free function** if f

- Is graded: If $X \in M_n(\mathbb{C})^d \cap D$, then $f(X) \in M_n(\mathbb{C})$.
- Respects direct sums: If $X, Y \in D$, then

$$f \begin{bmatrix} X & \\ & Y \end{bmatrix} = \begin{bmatrix} f(X) & \\ & f(Y) \end{bmatrix}.$$

- Respects similarity: If S is $n \times n$ & invertible with $X, S^{-1}XS \in D$, then

$$f(S^{-1}XS) = S^{-1}f(X)S.$$

Ex 1. Every non-commutative polynomial $p \in \mathbb{C}[X_1, \ldots, X_d]$ is a free function on every free set D.

Ex 2. $f(X) = X_1^{1/2} \left(X_1^{-1/2} X_2 X_1^{-1/2} \right)^{1/2} X_1^{1/2}$ is a free function on the set of pairs (X_1, X_2) of positive semi-definite matrices

The NC Loewner's Theorem

The NC analogue of $\mathbb{H}^d$ is

$$\Pi^d = \{X \in W^d : \ \mathrm{Im} X_i = \tfrac{1}{2i}(X_i - X_i^*) > 0, i = 1, \ldots, d\}.$$

The **Free Pick class** is the set of free functions f that map Π^d into $\overline{\Pi^1}$.

The NC analogue of $\mathbb{R}^d$ is $R^d := \{X \in W^d : X_i = X_i^*, i = 1, \ldots, d\}$.

A **real convex domain** is a free set D in R^d such that for each n, $D \cap M_n(\mathbb{C})^d$ is convex and open.

A function f is **matrix monotone** on a real convex domain $D \subseteq R^d$ if

$$X \leq Y \text{ implies } f(X) \leq f(Y) \quad \text{for all } X, Y \in D.$$

Non-commutative Loewner's Theorem (Pascoe-Tully-Doyle 2017, Pascoe 2017)

Let D be a convex real domain.

A free function $f : D \to R$ is matrix monotone on D if and only if f extends to a function in the free Pick class continuous on $D \cup \Pi^d$.

Any questions?

Matrix Monotonicity in the Quasi-Rational Setting

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Joint work with J.E. Pascoe and Ryan Tully-Doyle

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My Collaborators!

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Based on our preprint: Analytic continuation of concrete realizations and the McCarthy Champagne conjecture

Loewner's Theorem

A function $f : E \to \mathbb{R}$ is matrix monotone on E if and only if f analytically continues to $\mathbb{H}$ as a map $f : \mathbb{H} \cup E \to \overline{\mathbb{H}}$ in the Pick class.

Question: What is the analogue in 2 (commuting) variables?

Let $E \subseteq \mathbb{R}^2$ be convex and let $f : E \to \mathbb{R}$. If $A = (A_1, A_2)$ is a pair of commuting self-adjoint $n \times n$ matrices, with joint spectrum $\sigma(A) \subseteq E$, then

$$A_j = U \begin{pmatrix} \lambda_1^j & & \\ & \ddots & \\ & & \lambda_n^j \end{pmatrix} U^*, \text{ then } f(A) = U \begin{pmatrix} f(\lambda_1^1, \lambda_1^2) & & \\ & \ddots & \\ & & f(\lambda_n^1, \lambda_n^2) \end{pmatrix} U^*.$$

f is **globally matrix monotone on E** if whenever $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are pairs of commuting $n \times n$ self-adjoint matrices with $\sigma(A), \sigma(B) \subseteq E$,

$$A_j \leq B_j \text{ for } j = 1, 2 \quad \text{implies} \quad f(A) \leq f(B).$$

Two-variable Loewner Theorem

f is **locally matrix monotone on E** if whenever $\gamma(t)$ is a C^1 path of commuting pairs of self-adjoint matrices with $\sigma(\gamma(t)) \subseteq E$ and $\gamma_i'(t) \geq 0$ for $i = 1, 2$, then

$$t_1 \leq t_2 \text{ implies that } f(\gamma(t_1)) \leq f(\gamma(t_2)).$$

Note: In 1-variable, local monotonicity implies global monotonicity. Assume $A \leq B$ and set $\gamma(t) = (1-t)A + tB$. Then $\gamma'(t) = B - A \geq 0$ and

$$f(A) = f(\gamma(0)) \leq f(\gamma(1)) = f(B).$$

In 2-variables, pairs of commuting matrices A and B cannot necessarily be connected by a curve of pairs of commuting self-adjoint matrices.

Agler-McCarthy-Young 2012, Pascoe 2019

A function $f : E \to \mathbb{R}$ is locally matrix monotone on E if and only if f analytically continues to $\mathbb{H}^2$ as a map $f : E \cup \mathbb{H}^2 \to \overline{\mathbb{H}}$ in the Pick class.

McCarthy Champagne Conjecture (MCC):

Every 2-variable Pick function that analytically continues across an open convex set $E \subseteq \mathbb{R}^2$ (and is real-valued there) is globally matrix monotone when restricted to E.

Agler-McCarthy-Young 2012

Let f be a rational function of two variables. Let Γ be the zero-set of the denominator of f. Assume f is real-valued on $\mathbb{R}^2 \setminus \Gamma$. Let E be an open rectangle in $\mathbb{R}^2 \setminus \Gamma$.
Then f is globally matrix monotone on E if and only if f analytically continues to $\mathbb{H}^2$ as a Pick function.

Recall: $\alpha : \mathbb{D} \to \mathbb{H}$ defined by $\alpha(z) = i\left(\frac{1+z}{1-z}\right)$.

Let $\phi = \alpha^{-1} \circ f \circ \alpha$. Then ϕ is rational and holomorphic on $\mathbb{D}^2$ and $|\phi(\tau)| = 1$ a.e. on $\mathbb{T}^2 = (\partial\mathbb{D})^2$. So, ϕ is inner and extends continuously to $\alpha^{-1}(E) \subseteq \mathbb{T}^2$.

Proof Idea.

- Identify a useful **transfer function realization** for ϕ.
- Transfer this realization to $\mathbb{H}^2$ to get a realization for f.
- Use additional/known results to conclude global matrix monotonicity.

Realization Review

Each Schur function $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ possess a **transfer function realization**, i.e.

$$\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad \text{for } z \in \mathbb{D}^2, \quad \text{where}$$

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix}$$

is a contraction on a Hilbert space $\mathbb{C} \oplus \mathcal{M}$. The operator U can be chosen to be isometric, coisometric, or unitary.

The **Realization Hilbert space** $\mathcal{M}$ decomposes as $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $E_z = z_1 P_1 + z_2 P_2$ where each P_j is the projection onto $\mathcal{M}_j$.

Ex. Let $\phi(z) = \dfrac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}$. Set

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1/2 & -1/2 \\ \sqrt{2}/2 & -1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad E_z = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}.$$

Then $\phi(z) = A + B(1 - E_z D)^{-1} E_z C$ for $z \in \mathbb{D}^2$.

Global Monotonicity for Rational Functions

(Ball-Sadosky-Vinnikov, 2005), (Knese, 2011): If ϕ is rational & inner with $\deg \phi = (m_1, m_2)$ then ϕ has a unitary transfer function realization with $\dim \mathcal{M}_1 = m_1$ and $\dim \mathcal{M}_2 = m_2$.

Agler-McCarthy-Young 2012

Let f be a rational function of two variables. Let Γ be the zero-set of the denominator of f. Assume f is real-valued on $\mathbb{R}^2 \setminus \Gamma$. Let E be an open rectangle in $\mathbb{R}^2 \setminus \Gamma$.
Then f is globally matrix monotone on E if and only if f analytically continues to $\mathbb{H}^2$ as a Pick function.

Let $\alpha(z) = i\left(\frac{1+z}{1-z}\right)$ and $\phi = \alpha^{-1} \circ f \circ \alpha$. (Assume that $\deg \phi = \deg \mathrm{denom}(\phi)$.)

- ϕ is rational and inner on $\mathbb{D}^2$ and extends continuously to $\alpha^{-1}(E) \subseteq \mathbb{T}^2$.
- ϕ has a minimal TRF: $\phi(z) = A + B(1 - E_z D)^{-1} E_z C$, for all $z \in \mathbb{D}^2$.
- $(1 - E_z D)^{-1}$ is defined on $\alpha^{-1}(E)$ and so, the TFR extends to $\alpha^{-1}(E)$.

Goal of Project

Prove the McCarthy Champagne Conjecture for a much larger class of functions

Hilbert Spaces for Inner Functions

Let $H^2(\mathbb{D}^2)$ denote the Hardy space on $\mathbb{D}^2$:

$$H^2 = H^2(\mathbb{D}^2) = \left\{ f \in \mathrm{Hol}(\mathbb{D}^2) : \|f\|_{H^2}^2 = \lim_{r \nearrow 1} \int_{\mathbb{T}^2} |f(r\tau)|^2 dm(\tau) \ < \infty \right\}.$$

Let M_{z_1} and M_{z_2} denote multiplication by z_1 and z_2 on H^2.

Let $\phi : \mathbb{D}^2 \to \mathbb{D}$ be inner, i.e. $\phi \in \mathrm{Hol}(\mathbb{D}^2)$ and $\lim_{r \nearrow 1} |\phi(r\tau)| = 1$ for a.e. $\tau \in \mathbb{T}^2$.

Then we can define some useful subspaces of H^2 using ϕ:

- ϕH^2 is a Hilbert subspace of H^2.
- $K_\phi = H^2 \ominus \phi H^2$
- $S_1^{max} =$ the maximal subspace of K_ϕ invariant under M_{z_1}
- $S_2^{min} = K_\phi \ominus S_1^{max}$ is invariant under multiplication by M_{z_2}

Define the **realization Hilbert spaces** of ϕ by

$$\mathcal{M}_1 = S_2^{min} \ominus M_{z_2} S_2^{min} \quad \text{and} \quad \mathcal{M}_2 = S_1^{max} \ominus M_{z_1} S_1^{max}.$$

Realizations for inner functions

Let $\phi : \mathbb{D}^2 \to \mathbb{D}$ be inner and recall

$$\mathcal{M}_1 = S_2^{min} \ominus M_{z_2} S_2^{min} \quad \text{and} \quad \mathcal{M}_2 = S_1^{max} \ominus M_{z_1} S_1^{max}.$$

Set $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and let P_j be the projection of H^2 onto $\mathcal{M}_j$. Define

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{M} \end{bmatrix} \quad \text{as follows:}$$

- $Ax = \phi(0)x$ for all $x \in \mathbb{C}$
- $Bf = f(0)$ for all $f \in \mathcal{M}$
- $Cx = \left(P_1 M_{z_1}^* \phi + P_2 M_{z_2}^* \phi\right) x$ for all $x \in \mathbb{C}$
- $Df = \left(P_1 M_{z_1}^* + P_2 M_{z_2}^*\right) f.$

Theorem 1 (B. Knese, 2016, B.-Pascoe-Tully-Doyle, 2020)

Then U is a coisometry and if $E_z = z_1 P_1 + z_2 P_2$, then

$$\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad \text{for } z \in \mathbb{D}^2.$$

See also, (Ball-Sadosky-Vinnikov, 2005), (B.-Knese 2013), (Ball-Bolotnikv 2012), (Ball-Sadosky-Vinnikov-Kaliuzhnyi-Verbovetskyi, 2015)

Quasi-Rational Functions

ϕ is **quasi-rational with respect to an open** $I \subseteq \mathbb{T}$ if ϕ is inner and extends continuously to $\mathbb{T} \times I$ with $|\phi(\tau)| = 1$ for $\tau \in \mathbb{T} \times I$.

Ex. Let $\theta(z) = \dfrac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}$. Let ψ be a one-variable inner function that omits the value 1 on some open $I \subseteq \mathbb{T}$. Then

$$\phi(z) := \theta\left(z_1, \psi(z_2)\right) \text{ is quasi-rational on } I.$$

- Compositions of rational inner functions with (fairly) general 1-variable inner functions are quasi-rational.
- Quasi-rational functions behave like rational functions in 1 of the variables ($\phi(\cdot, \tau_2)$ is a finite Blaschke product for $\tau_2 \in I$).
- The set of quasi-rational functions is closed with respect to finite products.
- If (ϕ_n) is a sequence of quasi-rational functions on I that converges to some function ϕ both in the $H^2(\mathbb{D}^2)$ norm and locally uniformly on $\mathbb{D}^2 \cup (\mathbb{T} \times I)$, the ϕ is quasi-rational on I.

Continuation of the Realization

Theorem 2 (B.-Pascoe-Tully-Doyle, 2020)

If ϕ is quasi-rational with respect to I, then $(1 - E_\tau D)^{-1}$ exists for all $\tau \in \mathbb{T} \times I$.

Theorem 2 implies

$$\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad \text{for } z \in \mathbb{D}^2 \cup (\mathbb{T} \times I)$$

Structure of the Proof

- Show that $(1 - E_\tau D)$ has dense range.
- Show that $(1 - E_\tau D)$ is bounded below.

Main Tools

- Behavior of $(1 - E_\tau D)$ on a dense set of $\mathcal{M}$.
- (B.-Knese '13). All $f \in \mathcal{M}$ extend to $\Omega \supseteq \mathbb{D}^2 \cup (\overline{\mathbb{D}} \times I)$ and point evaluation is a bounded linear functional on $\mathcal{M}$ for all $z \in \Omega$.
- $J_\tau : \mathcal{M}_1 \to H^2(\mathbb{D})$ defined by $(J_\tau f)(z) = f(z, \tau)$ is an isometry for all $\tau \in I$.

Main Theorem

ϕ is **quasi-rational with respect to an open $I \subseteq \mathbb{T}$** if ϕ is inner and extends continuously to $\mathbb{T} \times I$ with $|\phi(\tau)| = 1$ for $\tau \in \mathbb{T} \times I$.

Let $\alpha(z) = i\left(\frac{1+z}{1-z}\right) : \mathbb{D} \to \mathbb{H}$.

Theorem 3 (B.-Pascoe-Tully-Doyle, 2020)

Let ϕ be quasi-rational with respect to an open $I \subseteq \mathbb{T}$, and let $f = \alpha \circ \phi \circ \alpha^{-1}$.

Then f is globally matrix monotone on every open rectangle $E \subseteq \mathbb{R} \times \alpha(I)$ (as long as ϕ omits the value 1 on $\alpha^{-1}(E)$).

Proof Idea: Theorem 2 implies that

$$\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad \text{for } z \in \mathbb{D}^2 \cup (\mathbb{T} \times I).$$

- Using conformal maps to assume $E = (0, \infty)^2$ and $f(\infty, \infty) \in \mathbb{R}$.
- Transfer this realization to $\mathbb{H}^2$ to get a realization for f on $\mathbb{H}^2 \cup (0, \infty)^2$
- Use NC Loewner Theorem to conclude monotonicity.

Transferring Realizations

Corollary 1 (B.-Pascoe-Tully-Doyle, 2020)

Assume $\phi(z) = A + B(1 - E_z D)^{-1} E_z C$ for $z \in \Omega \cup \{(1,1)\}$ such that

- $\phi(z) \neq 1$ in Ω,
- $z_1, z_2 \neq 1$ for $z \in \Omega$,
- and the realization also holds at $(1,1)$ and $\phi(1,1) \neq 1$.

Let $f = \alpha \circ \phi \circ \alpha^{-1}$ and $T = i(1+U)(1-U)^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$. Then

$$f(w) = T_{11} - T_{12}(E_w + T_{22})^{-1} T_{21} \text{ for } w \in \alpha(\Omega).$$

For us, U is unitary, so T is self-adjoint. Then

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} a & \gamma^* \\ \gamma & A \end{bmatrix},$$

for $a \in \mathbb{R}$, $\gamma \in \mathcal{M}$, and A a self-adjoint operator on $\mathcal{M}$. So

$$f(w) = a - \left\langle (A + E_w)^{-1}\gamma, \gamma \right\rangle_{\mathcal{M}} \text{ for } w \in \mathbb{H}^2 \cup (0,\infty)^2.$$

Theorem 3 (B.-Pascoe-Tully-Doyle, 2020)

Let ϕ be quasi-rational with respect to an open $I \subseteq \mathbb{T}$, and let $f = \alpha \circ \phi \circ \alpha^{-1}$.

Then f is globally matrix monotone on every open rectangle $E \subseteq \mathbb{R} \times \alpha(I)$ (as long as f is well defined on E).

Proof: By Theorem 2, $\phi(z) = A + B(1 - E_z D)^{-1} E_z C \quad$ for $z \in \mathbb{D}^2 \cup (\mathbb{T} \times I)$.

Assume $E = (0, \infty)^2$. Then there exist $a \in \mathbb{R}, \gamma \in \mathcal{M}$ and self-adjoint A such that:

$$f(w) = a - \left\langle (A + E_w)^{-1} \gamma, \gamma \right\rangle_{\mathcal{M}} \text{ for } w \in \mathbb{H}^2 \cup (0, \infty)^2.$$

Because this holds on $(0, \infty)^2$, A must be positive semi-definite.

Then f extends to a free Pick function defined on Π^d by Pascoe-Tully-Doyle '17.

Since A is positive semi-definite, f extends to the real convex free domain $D \subseteq R^2$ consisting of pairs of positive definite matrices and maps it into R^1.

By the NC Loewner Theorem, f is matrix monotone on D in the NC sense.

Assume $A = (A_1, A_2), B = (B_1, B_2)$ are pairs of commuting self-adjoint $n \times n$ matrices such that $\sigma(A), \sigma(B) \in (0, \infty)^2$ and each $A_j \leq B_j$. Then $A, B \in D$ and so, $f(A) \leq f(B)$.

Thanks for listening!