MATH 644 Homework 1
Due 9/22 in class.
Beware of typographical errors.
It is to be assumed throughout the assignment that the dimension d is greater than 1.

When Ω is an open subset of $\mathbb{R}^d$, recall that we define a *distribution* on Ω to be any continuous linear functional on the space of C^∞ functions of compact support in Ω (which we call *test functions* on Ω for convenience), where we say that $\varphi_n \to \varphi$ in this topology when there is a compact set K containing the supports of each φ_n and of φ on which $\sup_{x \in K} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| \to 0$ as $n \to \infty$ for all multiindices α. For convenience, we also define

$$\varphi_r(x) := r^{-d} \varphi(r^{-1}x) \text{ and } \tau_{x_0}\varphi(x) := \varphi(x - x_0).$$

for any C^∞ function φ of compact support on $\mathbb{R}^d$, any positive real r, and any $x_0 \in \mathbb{R}^d$.

1. (Weakly Harmonic Implies Harmonic)

   (a) Suppose φ is a radial C^∞ function on $\mathbb{R}^d$. Show that there is an even C^∞ function f on $\mathbb{R}$ such that $\varphi(x) = f(|x|)$ for all $x \in \mathbb{R}$. [Hint: Pick any unit vector ω and let $f(t) := \frac{1}{2}\left(\varphi(t\omega) + \varphi(-t\omega)\right)$.] Conversely, for any even C^∞ function f on $\mathbb{R}$, f, show that $f(|x|)$ is C^∞ on $\mathbb{R}^d$.

   (b) Suppose φ is a radial C^∞ function of compact support on $\mathbb{R}^d$. Show that there is another radial C^∞ function of compact support ψ such that

   $$\Delta \psi(x) = d\varphi(x) + x \cdot \nabla \varphi(x).$$

   Hint: If $\varphi(x) := f(|x|)$ and $\psi(x) := g(|x|)$, take $g'(r) := rf(r)$. Use question 1 to show ψ is C^∞.

   (c) We say that a distribution F on Ω is weakly harmonic when $F(\Delta\varphi) = 0$ for all test functions φ. If F is weakly harmonic on Ω, show that $F(\tau_{x_0}\varphi_r)$ is well-defined and independent of r for all $x_0 \in \Omega$ and all r such that $B_r(x_0) \subset \Omega$. Show also that $x_0 \mapsto F(\tau_{x_0}\varphi_r)$ is a smooth function on its domain for each r.

   (d) If ψ is a test function on $\mathbb{R}^d$, show that

   $$\sum_{j \in \mathbb{Z}^d} \epsilon^d \psi(\epsilon j) \varphi_r(\cdot - \epsilon j) \to \psi * \varphi_r$$

   in the test function topology as $\epsilon \to 0^+$. Use this to show that when F is weakly harmonic on Ω,

   $$F(\varphi_r * \psi) = \int \psi(x) F(\tau_x \varphi_r) dx$$

   for all test functions ψ and all r sufficiently small. Then show that there exists $u \in C^\infty(\Omega)$ such that

   $$F(\psi) = \int_\Omega \psi(x) u(x) dx$$

   for all test functions ψ on Ω. Show that u has the property that

   $$\int u(x) \varphi_r(x - x_0) dx = u(x_0) \int \varphi(x) dx$$

   for all radial test functions φ, all $x_0 \in \Omega$, and all r sufficiently small. Conclude that u is harmonic.

2. (Schwarz Reflection Principle) Suppose that u is a harmonic function on the upper half ball

$$B_r^+(x) := \left\{ y \in \mathbb{R}^d \ \mid |y - x| < r \text{ and } y_d > x_d \right\}$$

and that u extends continuously to a function on the closure of $B_r^+(x)$ which vanishes everywhere on the hyperplane $y_d = x_d$. Show that u extends to a harmonic function on the whole ball $B_r(x)$.

Hint: It's easiest to show that u extends to a distribution on $B_r(x)$ which is weakly harmonic. When you extend u, take any test function φ on the ball and define $\varphi^\epsilon(y) := \eta(\epsilon^{-1}(y_d - x_d))\varphi(y)$, where η is some even smooth function on the real line which is identically one on $[-1,1]$ and vanishes outside $[-2,2]$. Then observe

$$\varphi(y) = \varphi^\epsilon(y) + \left[1 - \eta(\epsilon^{-1}(y_d - x_d))\right] \varphi(y).$$

Integrating the Laplacian of φ^ϵ against u is delicate: there is one term in which a factor of ϵ^{-2} appears and threatens to ruin everything. BUT the volume of the support of φ^ϵ is bounded by a constant times ϵ, and also $u \to 0$ on the support as $\epsilon \to 0$. If you also write

$$\int_{B_r(x)} u(y)\Delta\varphi^\epsilon(y)dy = \int_{B_r^+(x)} u(y)\left[\Delta\varphi^\epsilon(y) - \Delta\varphi^\epsilon(y')\right]dy$$

where y' is the reflection of y about the hyperplane $y_d = x_d$, then since η is even, the Mean Value Theorem gives you one final factor of ϵ, which is just enough to show that

$$\lim_{\epsilon \to 0^+} \int u(y)\Delta\varphi^\epsilon(y)dy = 0.$$

3. (Real Analyticity: Further Analysis)

   (a) Show that there is a function F on $D \times [-1,1]$, where D is the complex unit disk $\{z \in C \mid |z| < 1\}$ such that $F(z,w)$ is holomorphic in z and

   $$F(z,w) = (1 - 2wz + z^2)^{-\frac{d}{2}}$$

   when z is real.

   (b) Show that the coefficient of z^n in the Taylor series of $F(z,w)$ is a polynomial $p_n(w)$ of degree at most n in w and that p_n is even when n is even and p_n is odd when n is odd. Use the Cauchy Integral Formula for derivatives to show that for any $r \in (0,1)$, there is a constant C_r such that

   $$|p_n(w)| \leq C_r r^{-n}$$

   for all $w \in [-1,1]$.

   (c) Now suppose $f_y(x) := (1-|x|^2)|x-y|^{-d}$ when $x, y \in \mathbb{R}^d$ and $|y| = 1$ (note that $f_y(x)$ is harmonic on $B_1(0)$). Show that

   $$\lim_{n\to\infty} \sum_{|\alpha| \leq n} \frac{\partial^\alpha f_y(0)}{\alpha!} x^\alpha = f_y(x)$$

   for all $x \in \mathbb{R}^d$ with $|x| < 1$ with uniform convergence on compact subsets of $B_1(0)$. [Hint: $f_y(x) = (1-|x|^2)F(|x|, x \cdot y/|x|)$.]

   (d) Use the Poisson formula for harmonic functions on the ball to deduce that when u is harmonic on $B_r(x_0)$,

   $$u(x) = \lim_{n\to\infty} \sum_{|\alpha| \leq n} \frac{\partial^\alpha u(x_0)}{\alpha!}(x - x_0)^\alpha$$

   for all $x \in B_r(x_0)$ with uniform convergence on compact subsets of $B_r(x_0)$. In this sense, the Taylor series must converge on the largest possible ball.

(e) In part (c), you likely showed that the degree n part of the Taylor series of $|x-y|^{-d}$ equals

$$|x|^n p_n\left(\frac{x\cdot y}{|x|}\right) = \sum_{|\alpha|=n} c_\alpha(y) x^\alpha$$

for some constants $c_\alpha(y)$. Now show that

$$\sum_{|\alpha|=n} |c_\alpha(y)||x^\alpha| \geq |x|^n \left|p_n\left(\frac{ix\cdot y}{|x|}\right)\right|;$$

using this inequality, go back to part (a) and show that the radius of convergence of $F(z,i)$ is $\sqrt{2}-1<1$. Conclude that that the Taylor series of $f_y(x)$ at $x=0$ cannot be absolutely convergent when $|x|>\sqrt{2}-1$. Thus if we try to sum the Taylor series of $f_y(x)$ in some nonstandard order, it might not converge to $f_y(x)$ outside the smaller ball $|x|<\sqrt{2}-1$.

4. (Removable Singularities) Let $G(x,y)$ be the free space Green's function on $\mathbb{R}^d$, and suppose u is harmonic on $B_r(x_0)\setminus\{x_0\}\subset\mathbb{R}^d$ and that

$$\lim_{x\to x_0}\frac{|u(x)|}{|G(x,x_0)|}=0. \tag{†}$$

Show that u extends to a harmonic function on $B_r(x_0)$ (i.e., that giving $u(x_0)$ the appropriate value makes u harmonic there.) Hint: Apply the divergence theorem to

$$\int_{\Omega'}\left[G(x,y)\Delta u(y) - u(y)\Delta_y G(x,y)\right]dy$$

where the integral is taken over the domain Ω' which equals $B_r(x_0)$ minus two small balls centered at x and x_0, respectively. The case $d=2$ is the hardest. One option which works in all dimensions is to use derivative estimates to show that $|x-x_0|^d|\nabla u(x)|\to 0$ as $x\to x_0$ and then show that

$$\left|\int_{\partial B_\epsilon(x_0)} G(x,y)\left[\hat{n}\cdot\nabla u(y)\right]d\sigma(y) - G(x,x_0)\int_{\partial B_\epsilon(x_0)}\left[\hat{n}\cdot\nabla u(y)\right]d\sigma(y)\right|\to 0$$

as $\epsilon\to 0^+$. Show that $\Delta u=0$ on $B_r(x_0)\setminus\{x_0\}$ implies that the flux integral

$$\int_{\partial B_\epsilon(x_0)}\left[\hat{n}\cdot\nabla u(y)\right]d\sigma(y)$$

must be independent of ϵ for all $\epsilon<r$. Use this to prove that there is some constant c such that

$$u(x)-cG(x,x_0)$$

on $B_r(x_0)\setminus\{x_0\}$ equals a function which is actually smooth on $B_r(x_0)$. Then the limit (†) forces $c=0$. Once u is known to be smooth, since $\Delta u=0$ away from x_0, it must also equal zero at x_0.

Another totally distinct approach would be to exploit the fact that

$$\varphi\mapsto\int_{B_r(x_0)} u(x)\Delta\varphi(x)dx \tag{‡}$$

is a distribution which is supported at the point $\{x_0\}$, which means it must be a finite linear combination of partial derivatives of the Dirac delta function. This approach actually allows for a full classification of all possible singularities of u at x_0, when it is known that $|x-x_0|^N|u(x)|\to 0$ for some N (you'd have to replace $\Delta\varphi(x)$ in (‡) by $\Delta\varphi(x)$ minus its degree N Taylor polynomial at x_0 to guarantee the integral is well-defined). The answer is that

$$u(x)=p(\nabla_x)G(x,x_0)+\tilde{u}(x)$$

for some polynomial p and some harmonic function $\tilde{u}$ on the full ball $B_r(x_0)$. This generalizes the notion of a pole of a meromorphic function in the complex plane.