§2.1–2.3 Main Topics

- **Matrices**: notation, row and column vectors, transposition, diagonal matrices, traces, symmetric and antisymmetric matrices; matrix of coefficients $A$ and the augmented matrix $A\#$ of systems of linear equations

- **Matrix Algebra**: addition and multiplication, scalar multiplication of matrices; connections to systems of equations; Kronecker delta symbol; algebraic laws for matrices (addition, multiplication, and transposition)

- **Systems of Linear Equations**: Unknowns, coefficients, homogeneous versus inhomogeneous, solutions and solution sets, consistent versus inconsistent
• A **matrix** is any rectangular array of items (numbers, functions, etc.) The items themselves are called **entries** or **elements** of the matrix. A matrix with \( n \) rows and \( m \) columns is said to have size \( n \times m \) (read: \( n \) by \( m \))—i.e., rows always come first.

• Entries of a matrix are labelled with a pair of subscripts: for a matrix \( A \), the entry in row \( i \) and column \( j \) is usually denoted \( a_{ij} \) or \( A_{ij} \) (again, for consistency, the row index always comes first).

• An example:

\[
A := \begin{bmatrix}
4 & x & 0 & -2 \\
\pi & -3 & 1 & 1 \\
2 & \pi^2 & 3 & x^2
\end{bmatrix}
\]

Here \( A \) is a \( 3 \times 4 \) matrix; \( A_{23} = 1 \) and \( A_{32} = \pi^2 \).

• Two matrices are equal when they have the same size and all corresponding entries agree.
• Any matrix can be multiplied by a constant or a function through the operation called **scalar multiplication**. Just like vectors, you multiply every entry by the same factor:

\[
A := \begin{bmatrix}
4 & x & 0 & -2 \\
\pi & -3 & 1 & 1 \\
2 & \pi^2 & 3 & x^2 \\
\end{bmatrix}, \quad 3A = \begin{bmatrix}
12 & 3x & 0 & -6 \\
3\pi & -9 & 3 & 3 \\
6 & 3\pi^2 & 9 & 3x^2 \\
\end{bmatrix}
\]

In formulas: \((kA)_{ij} = k(A_{ij})\).

• If the matrices \(A\) and \(B\) have the same size, they may be added together. The formula for **matrix addition** is \((A + B)_{ij} := A_{ij} + B_{ij}\). Informally, corresponding entries sum:

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{bmatrix} + \begin{bmatrix}
9 & 7 & 5 \\
8 & 6 & 4 \\
\end{bmatrix} = \begin{bmatrix}
10 & 9 & 8 \\
12 & 11 & 10 \\
\end{bmatrix}.
\]

**Matrix addition is commutative:** \(A + B = B + A\).
Matrices of compatible sizes may be multiplied: We define the matrix product $AB$ when the number of columns of $A$ matches the number of rows of $B$.

- If $A$ is $n \times m$ and $B$ is $m \times p$, then the product $AB$ will be a matrix of size $n \times p$.
- In formulas: $(AB)_{ik} := \sum_{j=1}^{m} A_{ij}B_{jk}$. In other words: the $ik$-entry of $AB$ is computed by taking a dot product of row $i$ of matrix $A$ with column $k$ of matrix $B$.
- An example:

\[
\begin{bmatrix}
3 & x & 0 \\
-4 & x^2 & 1 \\
2 & x^3 & 0 \\
1 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 & -2 \\
3 & 1 \\
-3 & 9 \\
\end{bmatrix}
= 
\begin{bmatrix}
3x + 6 & x - 6 \\
3x^2 - 11 & x^2 + 17 \\
3x^3 + 4 & x^3 - 4 \\
-4 & 6 \\
\end{bmatrix}
\]

Matrix multiplication is not commutative!
Multiplication Exercise

\[
\begin{bmatrix}
4 & 4 & -1 & -1 \\
4 & -4 & -1 & 1 \\
-2 & -1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 2 & -2 & 6 \\
1 & 1 & 4 & 4 & 4 \\
1 & -1 & 8 & -8 & 24
\end{bmatrix}
\]

- Is this multiplication well-defined? What will be the dimensions of the product?
- Compute the entries.
Why this definition of multiplication?

1. Economy of notation for systems of linear equations:

\[
\begin{align*}
3x + 2y + z &= 8 \\
6x + 5y + 3z &= 0 \\
2x + 9y + z &= 1
\end{align*} \iff
\begin{bmatrix}
3 & 2 & 1 \\
6 & 5 & 3 \\
2 & 9 & 1
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix}
= \begin{bmatrix}
8 \\ 0 \\ 1
\end{bmatrix}.
\]

2. Compatibility with geometric operations like rotation: rotation by angle \(\theta\) in the plane is accomplished via multiplication of \(2 \times 1\) matrices and the matrix

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]

Angle addition formulas have a nice structure:

\[
\begin{bmatrix}
\cos(\theta + \phi) & -\sin(\theta + \phi) \\
\sin(\theta + \phi) & \cos(\theta + \phi)
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}.
\]
Laws of Matrix Algebra

Associative Laws:

\[ A + (B + C) = (A + B) + C \quad \text{(addition)} \]
\[ k_1(k_2A) = (k_1k_2)A \quad \text{(scalar mult.)} \]
\[ A(BC) = (AB)C \quad \text{(matrix mult.)} \]

Distributive Laws:

\[ k(A + B) = kA + kB \]
\[ (k_1 + k_2)A = k_1A + k_2A \]
\[ A(B + C) = AB + AC \]
\[ (A + B)C = AC + BC \]
The transpose of the $n \times m$ matrix $A$ is denoted $A^T$ and refers to the $m \times n$ matrix where rows and columns have been interchanged: $A_{ij}^T = A_{ji}$. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$ 

Laws of Transposition:

$$(A^T)^T = A$$
$$(A + B)^T = B^T + A^T$$
$$(AB)^T = B^T A^T$$
$$(kA)^T = k(A^T)$$

Note: The order of multiplication gets reversed when you transpose!
There’s lots of terminology to learn: A **square matrix** is one with equal numbers of rows and columns; a **column vector** is a matrix with only one column; a **row vector** is a matrix with only one row. **Upper Triangular** matrices have $A_{ij} = 0$ when $i > j$. **Lower Triangular** matrices have $A_{ij} = 0$ when $i < j$ (note that the book gets the definitions backwards!). **Diagonal** means that all entries $A_{ij}$ are zero when $i \neq j$ (so any nonzero entries must reside on the diagonal from top left to bottom right). **Identity** means that the matrix is square and has ones on the diagonal and zeros elsewhere. **Symmetric** means that it equals its own transpose. **Skew Symmetric** means that it is equal to the opposite of its own transpose.
Systems of Linear Equations

A system of \( n \) linear equations in \( m \) unknowns has the form:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1(m-1)}x_{m-1} + a_{1m}x_m &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2(m-1)}x_{m-1} + a_{2m}x_m &= b_2 \\
  &\vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n(m-1)}x_{m-1} + a_{nm}x_m &= b_n
\end{align*}
\]

If \( b_1 = b_2 = \cdots = b_n = 0 \), the system is called \textbf{homogeneous} (otherwise it is called \textbf{inhomogeneous}). A \textbf{solution} is a vector \( \vec{x} := \langle x_1, \ldots, x_m \rangle \) whose coordinates satisfy all the equations simultaneously.

**Fundamental Fact**

Any system of linear equations has either no solutions (called an inconsistent system), exactly one solution (called a unique solution), or infinitely many solutions.
Augmented Matrices

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1(m-1)}x_{m-1} + a_{1m}x_m = b_1 \]

\[ \vdots \]

\[ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n(m-1)}x_{m-1} + a_{nm}x_m = b_n \]

is encoded by the **augmented matrix** of coefficients:

\[
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} & b_1 \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{n1} & \cdots & a_{nm} & b_n
\end{bmatrix}
\]

- Given an augmented matrix, you can recover the system.
- What happens when you multiply the augmented matrix by

\[
\begin{bmatrix}
    x_1 \\
    \vdots \\
    x_m \\
    -1
\end{bmatrix}
\]

§2.4–2.6 Main Topics

- **Elementary row operations**: Manipulations of systems of equations which preserve the solution set.
- **Gaussian elimination**: Using Elementary row operations to solve systems of linear equations.
- **Matrix Inversion**: For solving “nice” systems.
Elementary Row Operations

- $P_{ij}$: Permute (i.e., interchange) row $i$ and row $j$

\[
\begin{bmatrix}
-1 & 0 & 1 \\
2 & 3 & 4 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & -1 \\
2 & 3 & 4 \\
-1 & 0 & 1
\end{bmatrix}
\]

- $M_i(\alpha)$: Multiply row $i$ by the constant $\alpha \neq 0$.

\[
\begin{bmatrix}
-1 & 0 & 1 \\
2 & 3 & 4 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 1 \\
1 & \frac{3}{2} & 2 \\
0 & 1 & -1
\end{bmatrix}
\]

- $A_{ij}(\alpha)$: Add $\alpha$ times row $i$ to row $j$.

\[
\begin{bmatrix}
-1 & 0 & 1 \\
2 & 3 & 4 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
5 & 9 & 13 \\
2 & 3 & 4 \\
0 & 1 & -1
\end{bmatrix}
\]

- Elementary row operations can also be achieved by multiplication on the left by a corresponding elementary matrix.
Identify what elementary row operations these matrices produce when multiplied on the left of an arbitrary matrix.

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -7 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

- What elementary matrix would permute row 2 and row 4 in a matrix with 6 rows?
- What elementary matrix would multiply row 3 by $-5$ in a matrix with 5 rows?
- What elementary matrix would add 3 times row 4 to row 2 in a matrix with 4 rows?
Row Reduction Exercise

When you are working on a problem in which row reduction is a sensible thing to do, you might want to give it a try if having more zeros in your matrix would make your problem easier. Apply elementary row operations to the following matrix and see how many entries you can force to equal zero:

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & -1 & 2 \\
3 & 2 & 1 \\
\end{bmatrix}
\]
Row Echelon Form

- The first nonzero entry of each row is a one (or the row has only zero entries).
- No two rows begin in the same column, and rows are arranged so that they begin further to the right as you move down.
- Rows of all zeros (if any) go at the bottom. Three examples:

- Columns with a leading 1 (green boxes) are called **pivotal columns**. The green boxes are also called **pivotal positions**.
- The matrix is in **reduced row echelon form** if the entries above every pivotal position are also all zero.
- If your matrix of coefficients is in row echelon form, you can easily solve the system using **back substitution**.
Row Reduction

You are going to proceed by cooking up the row echelon form one row at a time, starting at the top and working your way down. Once a row is found, you are going to “freeze it” and effectively ignore it from that point forward.

1. Find the leftmost column which is nonzero (ignore rows, if any, which are frozen). This will be a pivotal column.
2. Permute unfrozen rows so that the top unfrozen entry in your pivotal column is not a zero.
3. Multiply the top unfrozen row by an appropriate constant so that it has a 1 in the pivotal position. (Can be tricky with complex matrices.)
4. Apply elementary row operations $A_{ij}(\alpha)$ to all the subsequent rows so that you systematically obtain all zero entries below the pivotal position.
5. Freeze the row you just worked on.
6. Repeat these steps on the unfrozen rows as necessary.

The **Rank** of a matrix is the # of all zero rows in its REF.
Reduced Row Echelon Form is when every pivotal column of a row echelon matrix has zeros everywhere except the pivotal position. You can always put a row echelon matrix in reduced row echelon form by adding multiples of rows below to rows above (e.g., add a multiple of row 2 to row 1). Example pictures:

The reduced row echelon form of any matrix $A$ is unique, meaning that any two people who row reduce $A$ until they arrive at a RREF matrix (no matter what procedure they use) must ultimately get the same answer. This is not always true for non-reduced row echelon matrices.
1. Take the system of equations and form the Augmented Matrix.
2. Row reduce until you are in reduced row echelon form.
3. Convert the resulting matrix back into a system of equations.
4. Variables corresponding to pivotal columns are called **bound** because their values are forced.
5. Variables from non-pivotal columns (if any) are called **free** since their values could be anything.
6. If you end up with equations of the form $0 = 0$ at the end, just ignore. If you end up with $0 = 1$, this means the equations are inconsistent.
Example 1: Inconsistent Systems

\[
\begin{align*}
    x_1 & + & x_2 & + & x_3 & + & 3x_4 & = & 2 \\
    2x_1 & + & x_2 & + & 2x_3 & + & 5x_4 & = & 3 \\
    x_1 & - & x_2 & + & x_3 & + & x_4 & = & 1
\end{align*}
\]

This system of equations has no solutions. You might say that the solution set is the empty set: \(\emptyset\). Systems of any size can be inconsistent. Formally, a system will be inconsistent exactly when the rank of the matrix of coefficients is smaller than the rank of the augmented matrix. This happens exactly when the augmented matrix can be row reduced to contain a row in which the very last entry is pivotal.
Example 2: Underdetermined Systems

\begin{align*}
    x_1 & + x_2 + x_3 + 3x_4 = 2 \\
    2x_1 & + x_2 + 2x_3 + 5x_4 = 3 \\
    x_1 & - x_2 + x_3 + x_4 = 0
\end{align*}

Solution Set:

\[\{(1 - s - 2t, 1 - t, t, s) \mid t, s \in \mathbb{R}\}\]

The scalars \(s\) and \(t\) correspond to free variables \(x_3\) and \(x_4\). The number of bound variables is always the rank of the matrix of coefficients (\textbf{not} the rank of the augmented matrix). The number of free variables is the total number of variables minus the bound ones.
Example 3: Unique Solutions

\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 1 \\
    2x_1 + 2x_3 &= 0 \\
    -x_1 + 2x_2 + x_3 - 2x_4 &= -4 \\
    x_2 - x_4 &= -1
\end{align*}
\]

Unique Solution:

\[(1, 0, -1, 1)\]

Unique Solutions occur exactly when the matrix of coefficients (not the augmented matrix) has rank equal to the number of variables. If the number of equations equals the number of variables, this is the same as saying that the matrix of coefficients can be row-reduced to the identity matrix. If the number of equations is less than the number of variables, the solution is never unique.
Determine whether the following system is inconsistent, has a unique solution, or has infinitely many solutions. Then give a complete description of the solution set.

\[
\begin{align*}
y_1 &+ 2y_2 + y_3 + y_4 = 1 \\
y_1 &+ 2y_2 - y_3 - y_4 = 1 \\
3y_1 &+ 6y_2 - y_3 + y_4 = 1 \\
2y_1 &+ 4y_2 + 2y_3 + 3y_4 = 1
\end{align*}
\]
Balancing Chemical Equations

\[ \square \text{C}_5\text{H}_8 + \square \text{O}_2 \rightarrow \square \text{CO}_2 + \square \text{H}_2\text{O} \]

The unknowns are the coefficients; the equations come from atomic conservation (each element should appear the same number of times on each side).

Physical Statics

Vector Forces should sum to zero at every point.

[Diagram of a force diagram with angles and forces indicated]
At any point, incoming current equals outgoing current (current flows from positive to negative by convention). The sum of potential differences (voltage drops) around any closed loop equals zero; use $V = IR$ and add a minus sign if you cross the resistor in the same direction as the assumed current (no minus sign if going the opposite direction).
Inverse Matrices: §2.6

• Every square matrix $A$ of full rank has an inverse matrix $A^{-1}$ with the property that

$$A^{-1}A = AA^{-1} = I.$$ 

• To find $A^{-1}$ by the Gauss-Jordan technique: augment $A$ on the right with an identity matrix. Row reduce until the left half of the augmented matrix is an identity. The right half will be $A^{-1}$.

• The inverse matrix can be used to efficiently solve systems $Ax = b$. There are also several important algebraic properties of matrix inversion to keep in mind.
Matrix Inversion Exercises

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{2} & 1 & -\frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{2} & 1 & \frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]
Theorem

The following are all mathematically equivalent statements about the square matrix $A$:

1. $A$ is singular (i.e., it does not have an inverse).
2. There is a column vector $X \neq 0$ for which $AX = 0$.
3. The row-echelon form of $A$ has at least one row of all zeros.
4. The reduced row-echelon form of $A$ is not an identity matrix.
5. The rank of $A$ is strictly less than the number of rows.
6. The determinant of $A$ equals zero.
7. The equation $AX = 0$ has infinitely many solutions when $X$ is an unknown column vector.
8. It is possible to find a column vector $B$ such that the equation $AX = B$ is inconsistent when $X$ is an unknown column vector.
Facts about Matrix Inverses

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<td>1</td>
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<tr>
<td>2</td>
<td>$(AB)^{-1} = B^{-1}A^{-1}$.</td>
</tr>
<tr>
<td>3</td>
<td>$(A^T)^{-1} = (A^{-1})^T$.</td>
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</table>
The **determinant** of a square matrix $A$ is a number which has important algebraic and geometric properties. You’ve probably seen it for $2 \times 2$ and $3 \times 3$ matrices; the formula gets increasingly more complicated as the size of the square matrix grows. We’ll start developing the formula now.

A **permutation** of the integers $1, \ldots, n$ is any ordering of these numbers. There are a total of $n!$ such permutations.

To each permutation, we may assign a **signature** $\sigma$ which is equal to $+1$ or $-1$. Permutations with signature $+1$ are called **even** and permutations with signature $-1$ are called odd. Rule: Count the number of **inversions**—the number of pairs which appear in the wrong order—and then set the signature $+1$ if the number of inversions is even and odd otherwise.
Formula for the Determinant

\[ \text{det } A = \sum_{\text{all permutations } p} \sigma_{p_1 p_2 \ldots p_n} a_{1p_1} a_{2p_2} \cdots a_{np_n} \]

Example: 2 × 2

\[ \text{det} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \]

Example: 3 × 3

\[ \text{det} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} \]

\[ - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \]

In general, this is a long way to compute the determinant.
Terms to compute in a $4 \times 4$ determinant

There are 24 permutations of $\{1, 2, 3, 4\}$:

\[
\begin{align*}
&\ a_{11}a_{22}a_{33}a_{44} & a_{12}a_{21}a_{33}a_{44} & a_{13}a_{22}a_{31}a_{44} & a_{14}a_{22}a_{33}a_{41} \\
&\ a_{11}a_{23}a_{32}a_{44} & a_{11}a_{24}a_{33}a_{42} & a_{11}a_{22}a_{34}a_{43} & a_{12}a_{21}a_{34}a_{43} \\
&\ a_{13}a_{24}a_{31}a_{42} & a_{14}a_{23}a_{32}a_{41} & a_{12}a_{23}a_{31}a_{44} & a_{13}a_{21}a_{32}a_{44} \\
&\ a_{12}a_{24}a_{33}a_{41} & a_{14}a_{21}a_{33}a_{42} & a_{13}a_{22}a_{34}a_{41} & a_{14}a_{22}a_{31}a_{43} \\
&\ a_{11}a_{23}a_{34}a_{42} & a_{11}a_{24}a_{32}a_{43} & a_{12}a_{23}a_{34}a_{41} & a_{14}a_{21}a_{32}a_{43} \\
&\ a_{14}a_{23}a_{31}a_{42} & a_{13}a_{24}a_{32}a_{41} & a_{13}a_{21}a_{43}a_{42} & a_{12}a_{24}a_{31}a_{43} \\
\end{align*}
\]

There are 120 permutations of $\{1, 2, 3, 4, 5\}$. 
Method 2: Cofactors

1. Pick a row or a column to expand in. Say you pick row $i$. For each $j$, let $A(i|j)$ be the $(n-1) \times (n-1)$ matrix obtained by removing row $i$ and column $j$. It’s called the $(i, j)$-minor of $A$.

2. Take the determinant of the minors $A(i|j)$. The number $C_{ij} := (-1)^{i+j} \det A(i|j)$ is called the $(i, j)$-cofactor of $A$.

3. Compute the determinant of $A$ using the cofactors:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{or} \quad = \sum_{i=1}^{n} a_{ij} C_{ij}.$$ 

This is a recursive algorithm—to compute large determinants, you compute a large number of smaller determinants. It’s a little easier than the first method (in terms of bookkeeping), but it’s still somewhat lengthy.
Method 3: Row Reduction

The elementary row operations do nice predictable things to determinants:

1. Multiplying a row by a constant multiplies the determinant by the same constant.

2. Interchanging the order of two specific rows multiplies the determinant by $-1$.

3. Taking one row and adding a multiple of another row to it leaves the determinant alone.

Last key fact: The determinant of a triangular matrix is just the product of the diagonal entries. Using these facts, you can often deduce what the determinant should be in far fewer steps than the first method.
Other Properties of Determinants

• $\det A^T = \det A$.
• A matrix with a row (or column) of zeros has zero determinant.
• A matrix with two identical rows (or two identical columns) has zero determinant.
• The determinant gives the signed volume of parallelepipeds.

Already Mentioned:
• Interchanging two rows (or columns) changes the determinant by a factor of $-1$.
• Determinant of triangular is the product of diagonal entries.
• Multiplying a row by $k$ multiplies the determinant by $k$.
• Adding a multiple of a row to some other row leaves the determinant unchanged.

Final Major Property:
• $\det(AB) = (\det A)(\det B)$. 