How Big Can it Be? Some Challenges of Size in Fourier Analysis

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- The main part of the talk will be about the Kakeya Needle Problem, which examines whether sets which are large enough to move a needle-shaped object around in must also be large in the usual sense of area. This problem has an interesting and satisfying solution, but is also intimately connected to a host of open questions, large and small, in harmonic analysis.

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- As time permits, we will explore connections to geometric nonconcentration inequalities, which are a general framework for figuring out how to define largeness of sets so that it corresponds with whatever geometric properties that you find interesting.

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The Kakeya Needle Problem: Question

Among all such regions U, what is the smallest area of such a region?

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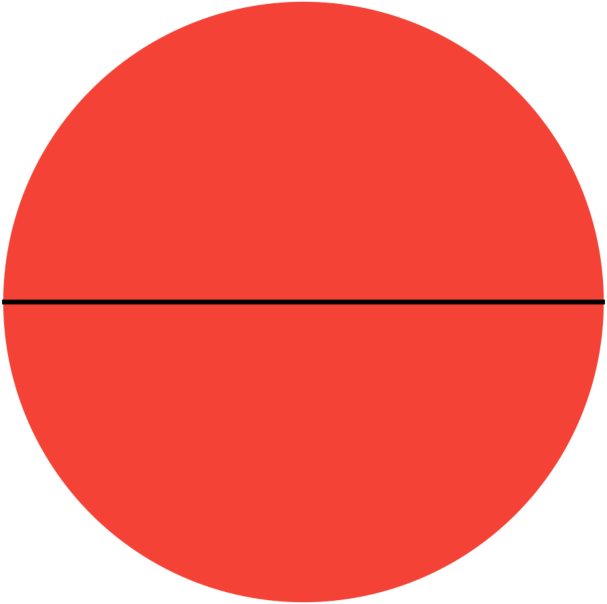

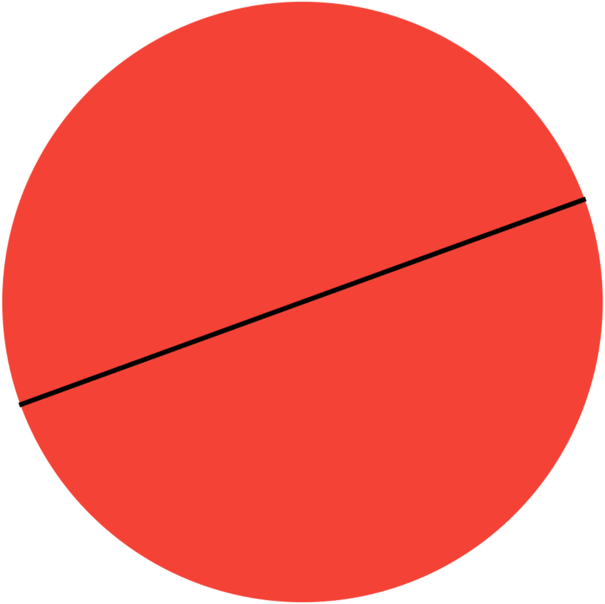

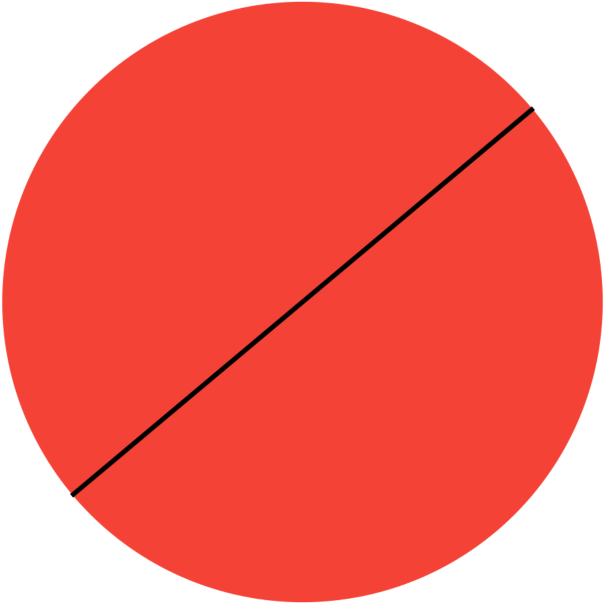

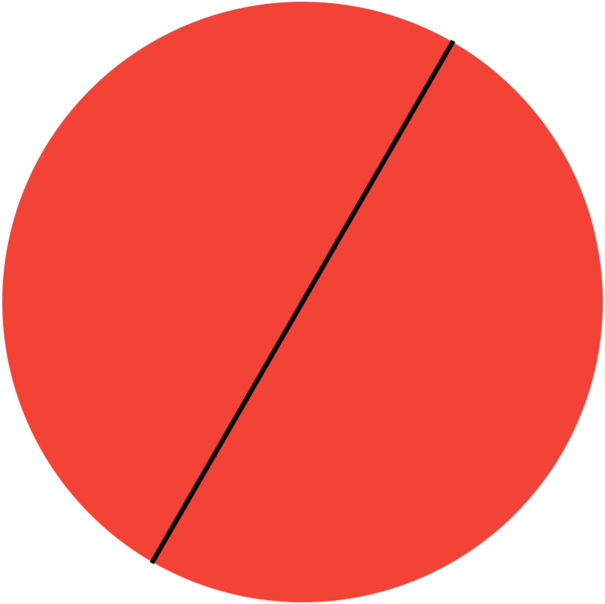

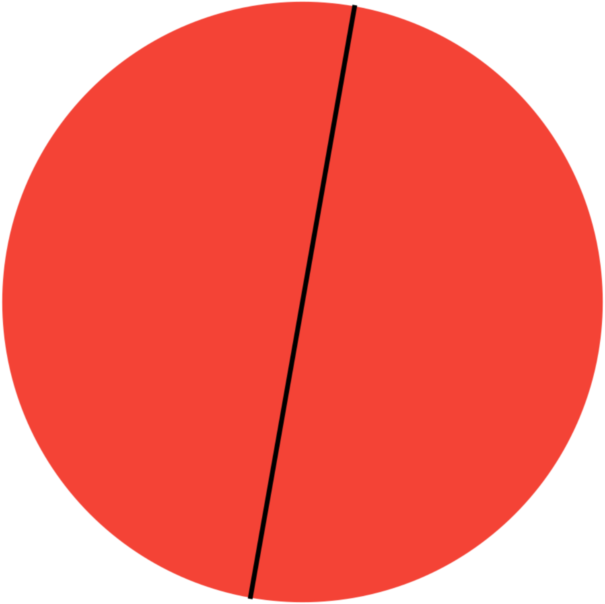

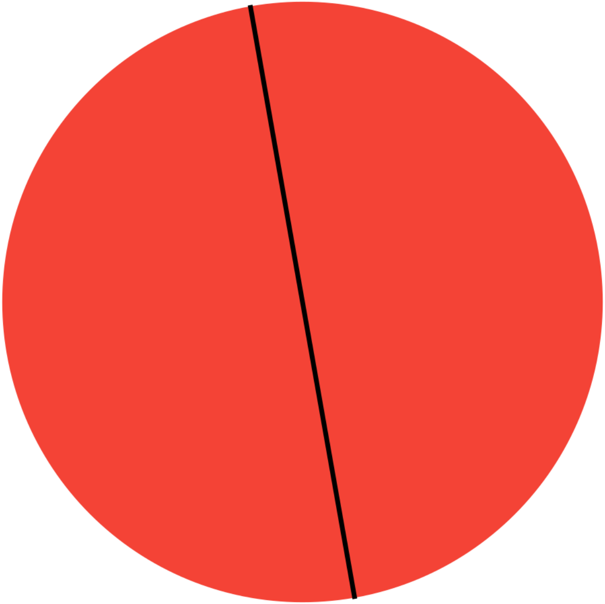

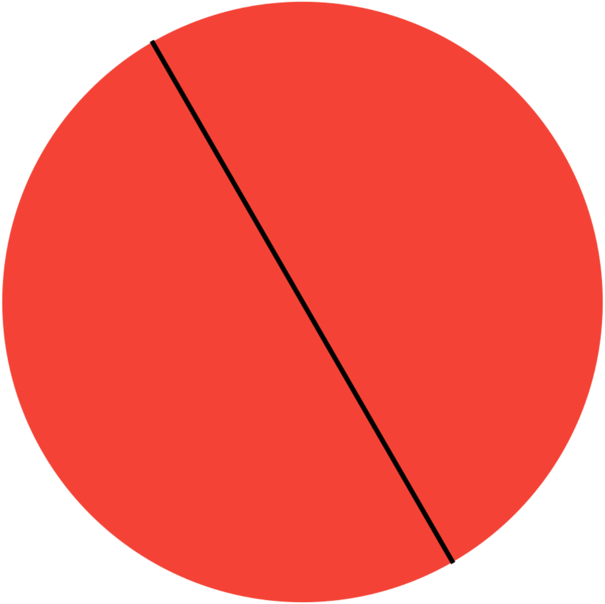

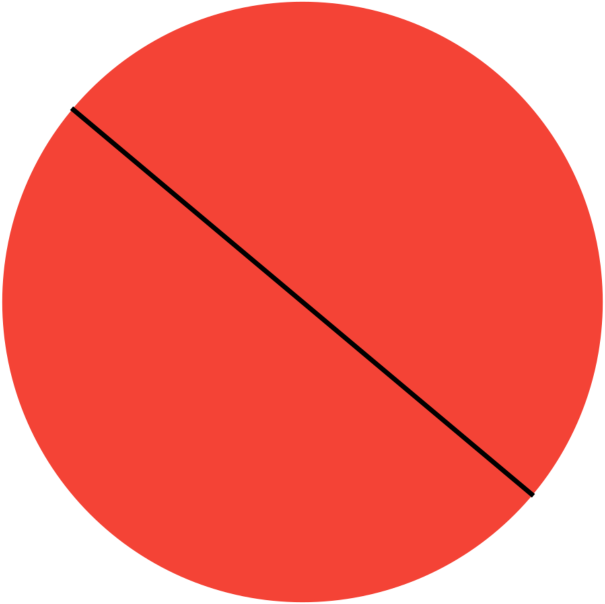

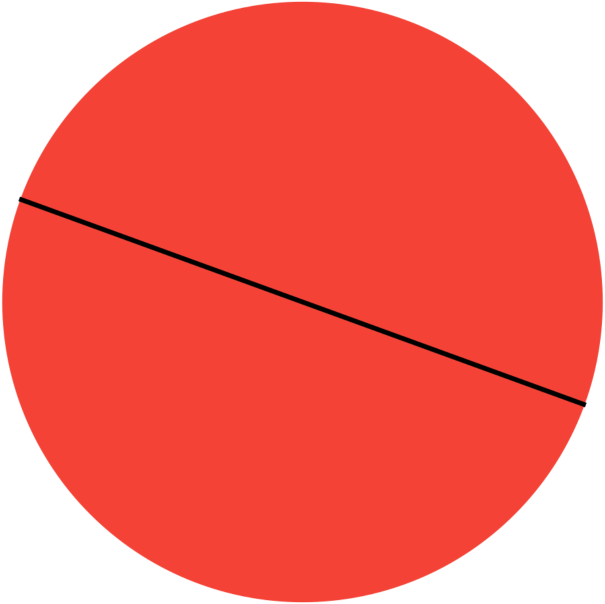

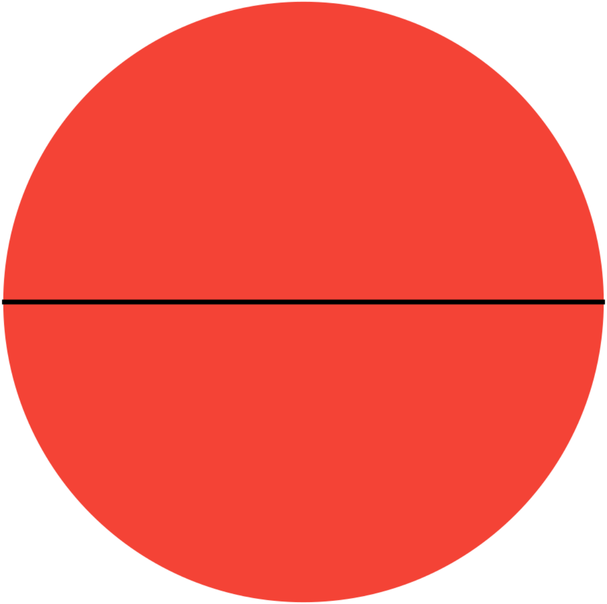

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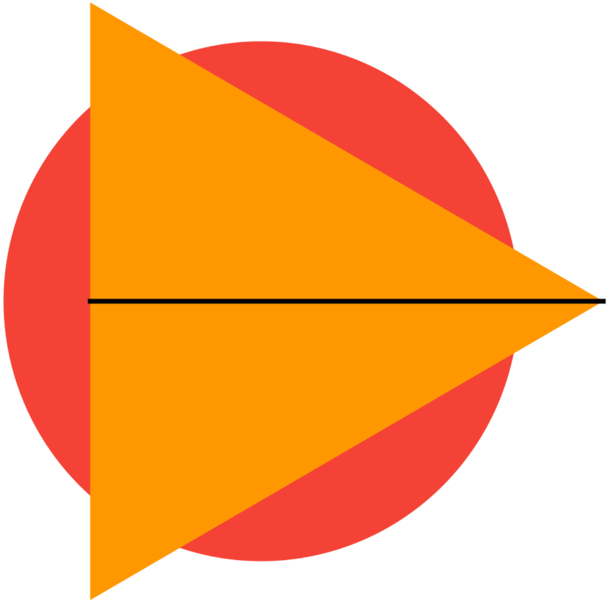

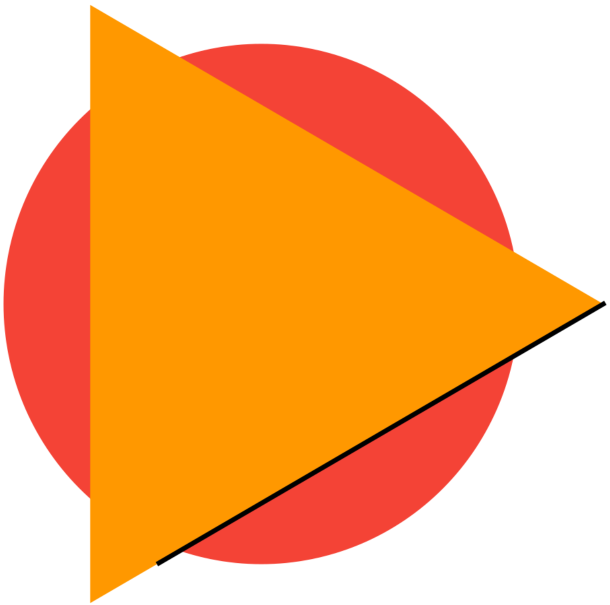

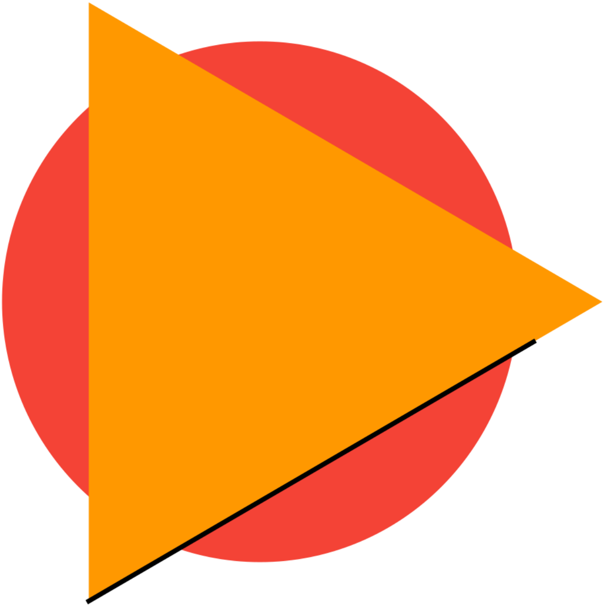

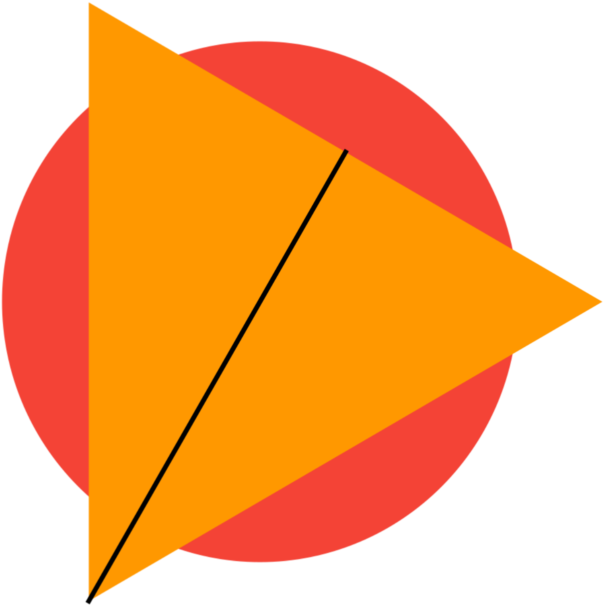

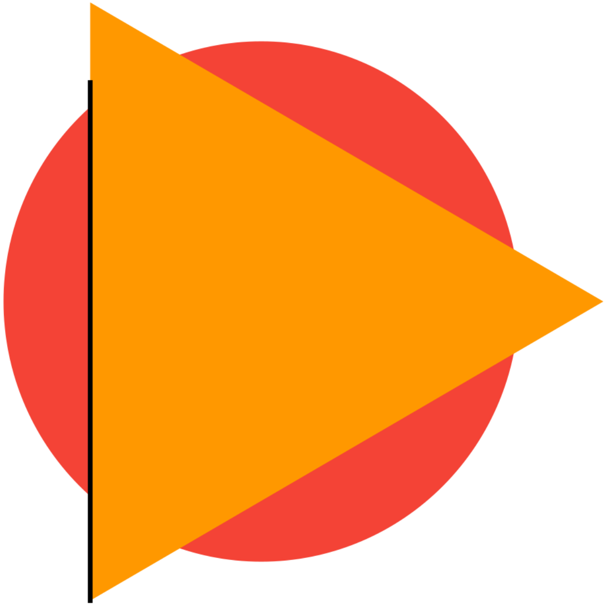

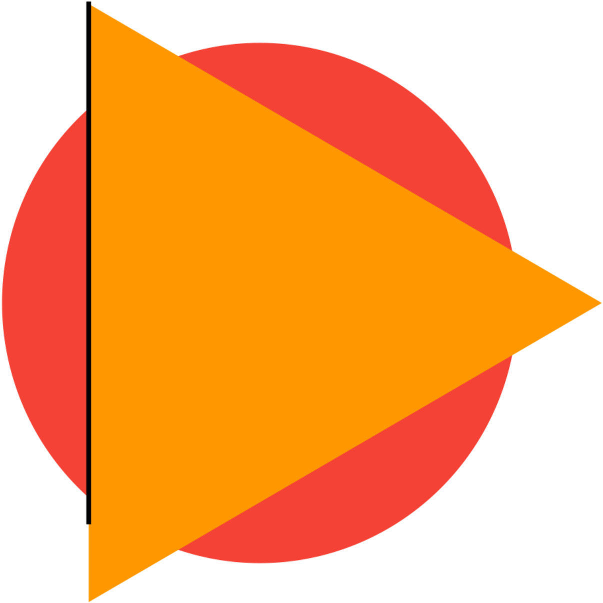

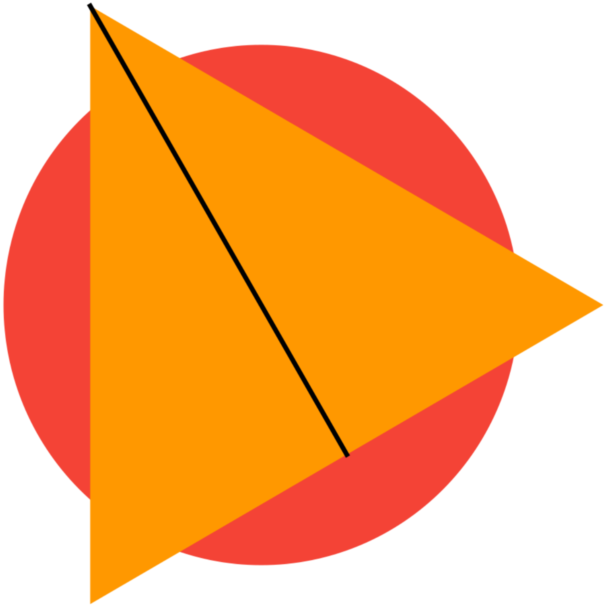

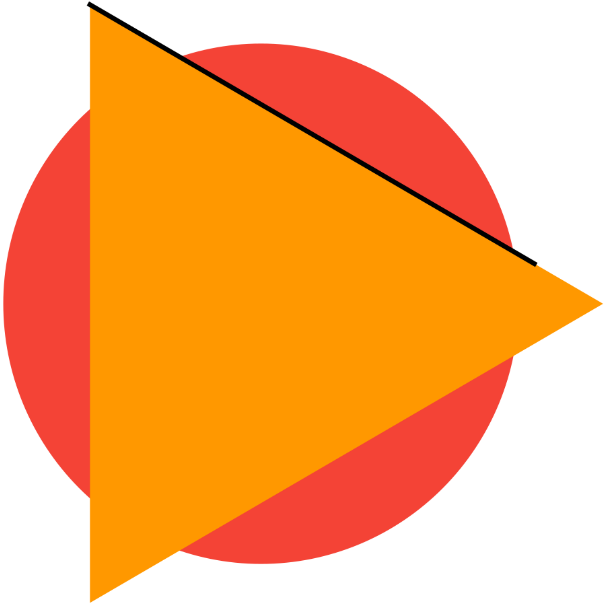

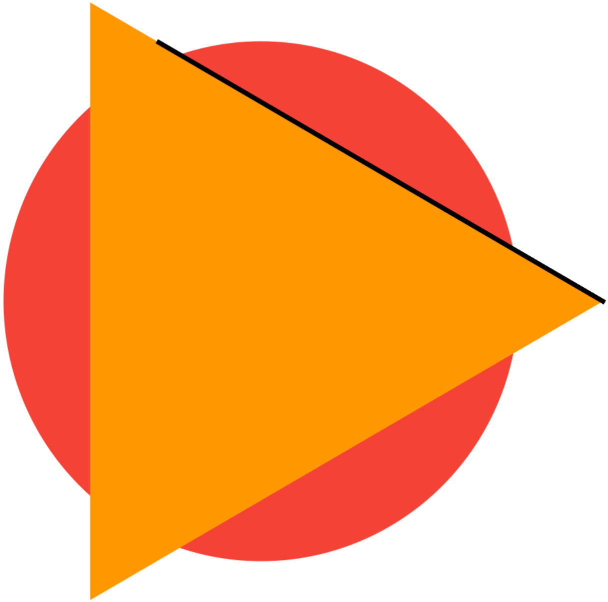

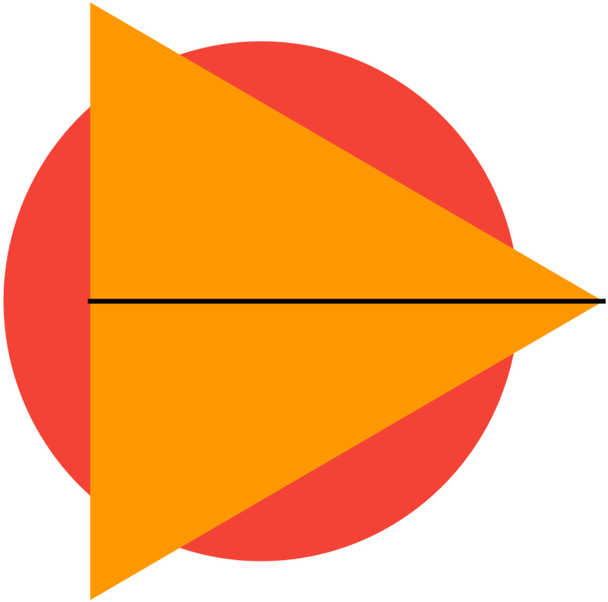

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E. Stein, <u>Harmonic Analysis</u>: "While this historical aspect [of the needle problem] has remained something of a curiosity, the Besicovitch set has come to play an increasingly significant role in real-variable theory and Fourier analysis. Indeed, our accumulated experience allows us to regard the structure of this set as, in many ways, representative of the complexities of two-dimensional sets, in the same sense that Cantor-like sets already display some of the typical features that arise in the one-dimensional case."

Connections to Kakeya

The Kakeya Conjecture. We define a Besicovitch set in $\mathbf{R}^n$ to be a set which contains a unit line segment in every direction. Such sets can have arbitrarily small (even zero) Lebesgue measure. Does the set have n-dimensional fractal measure ($n > 2$)?

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PDEs. The Kakeya conjecture is connected to regularity properties of PDEs. In particular, certain conjectured estimates on the regularity of solutions of the wave equation would imply Kakeya.

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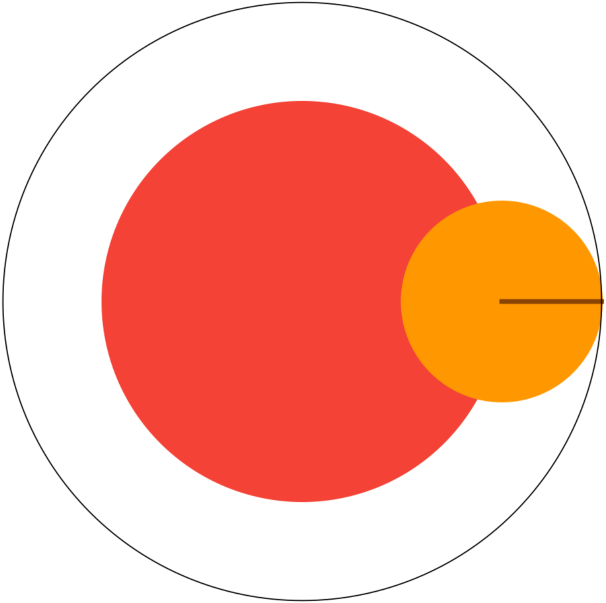

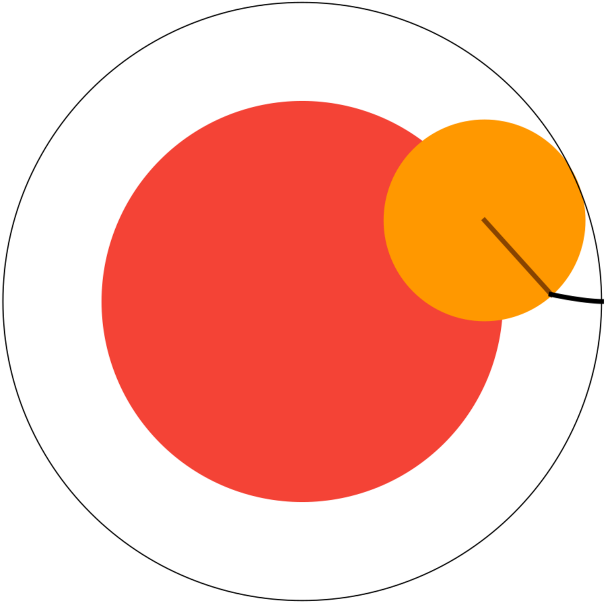

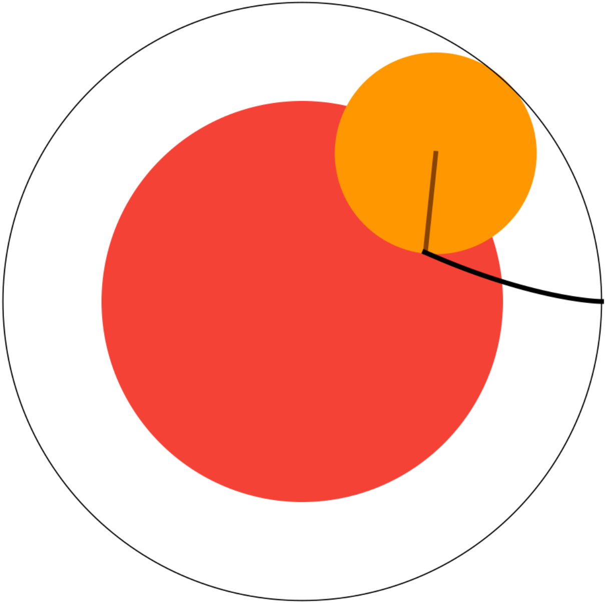

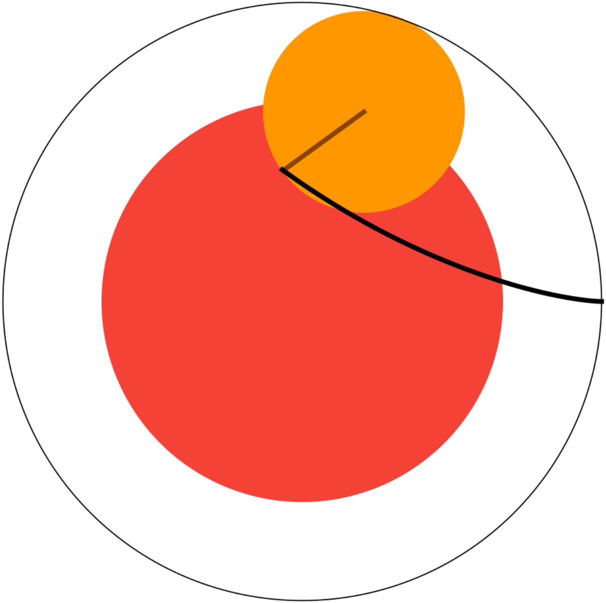

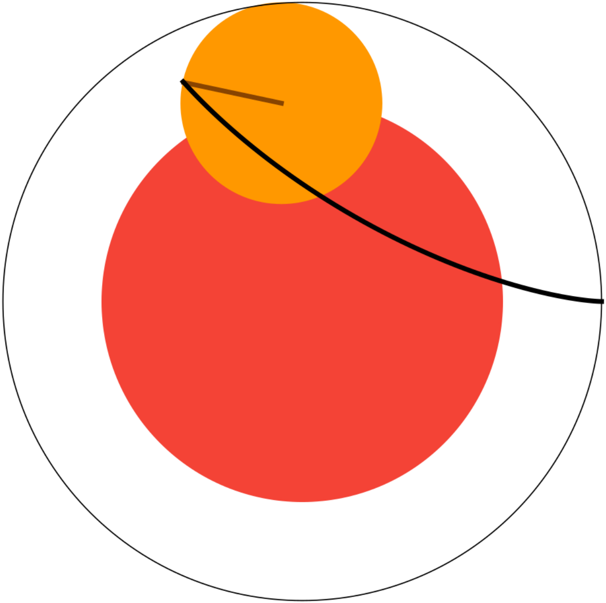

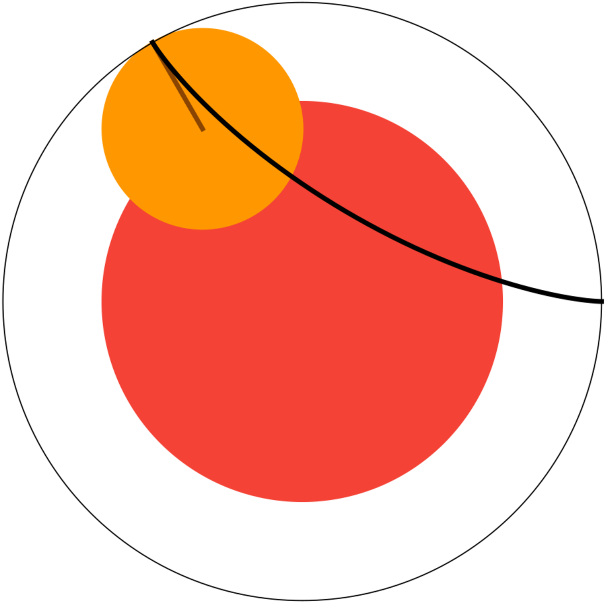

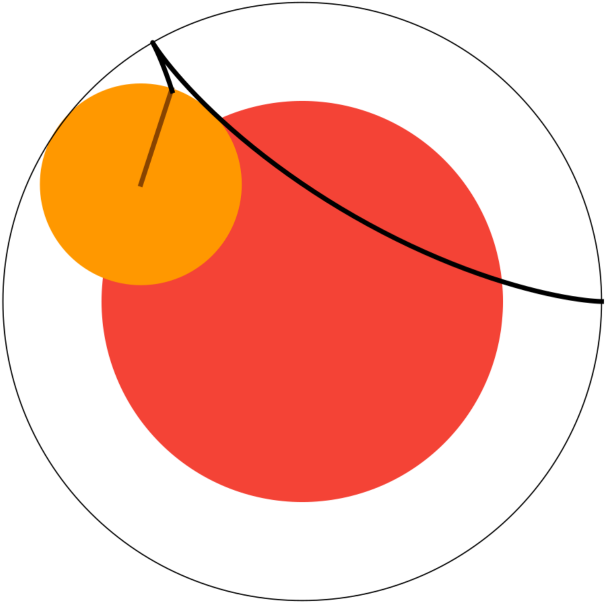

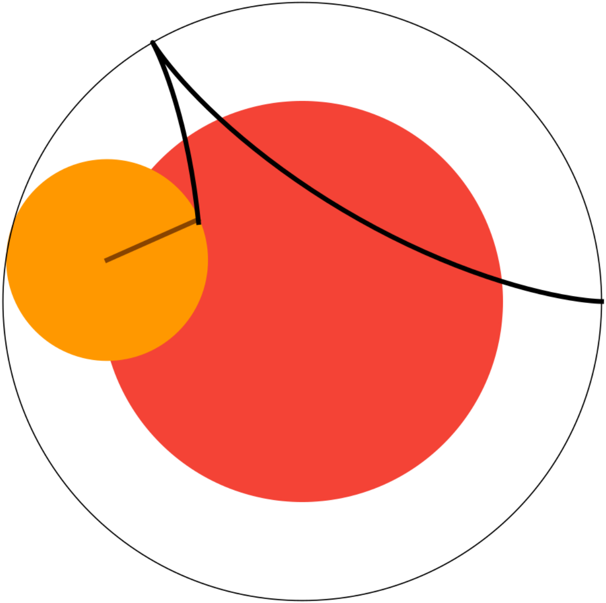

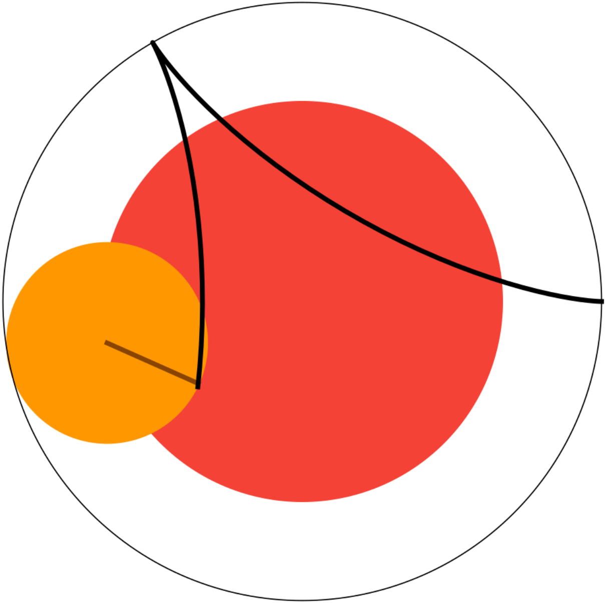

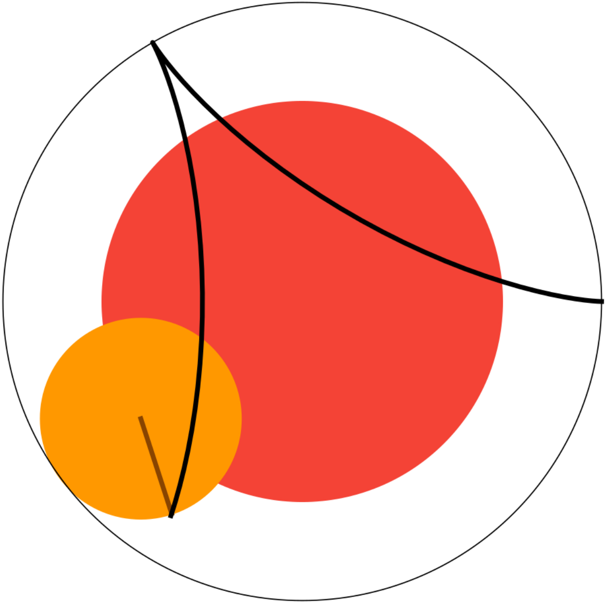

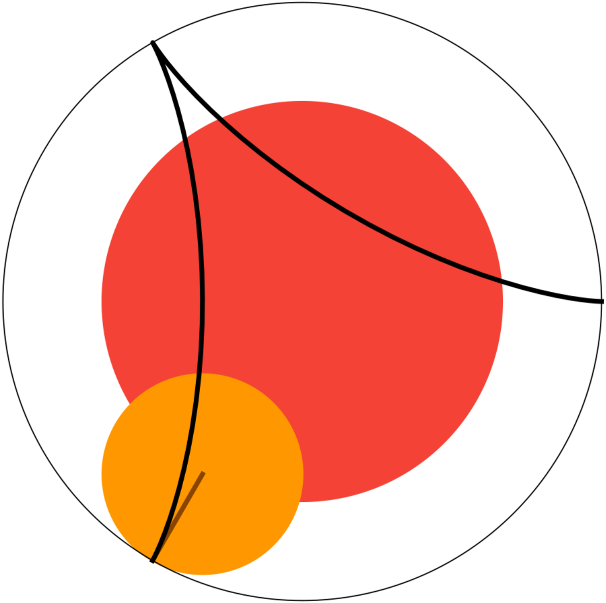

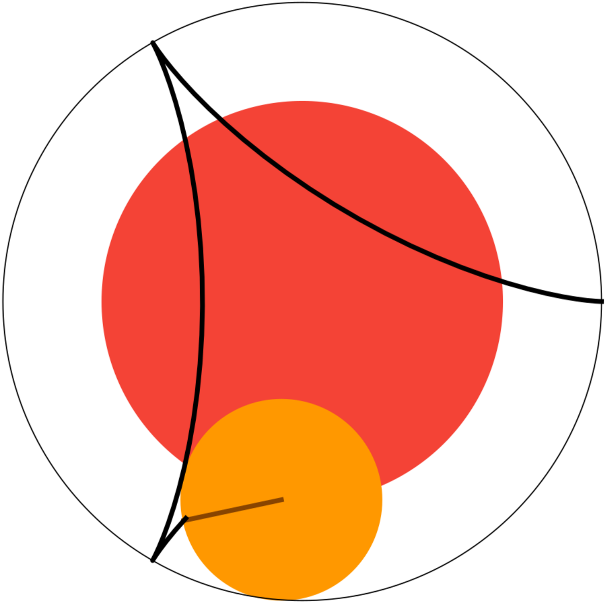

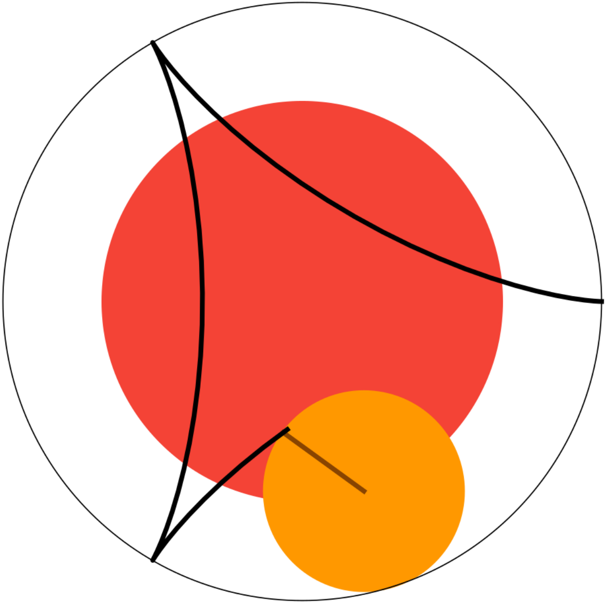

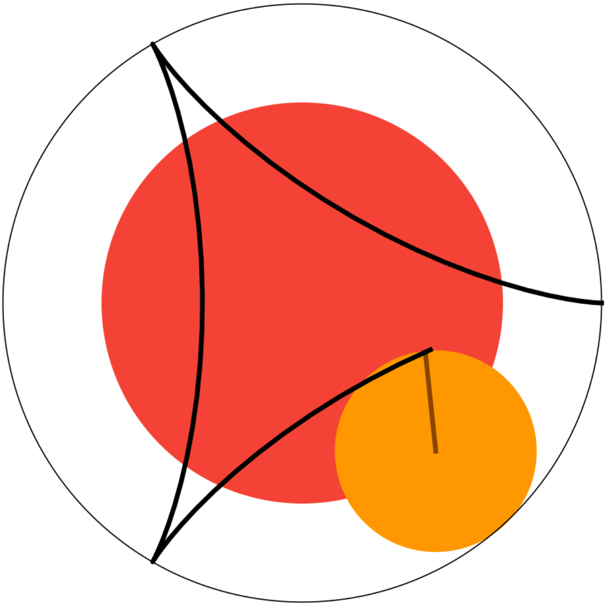

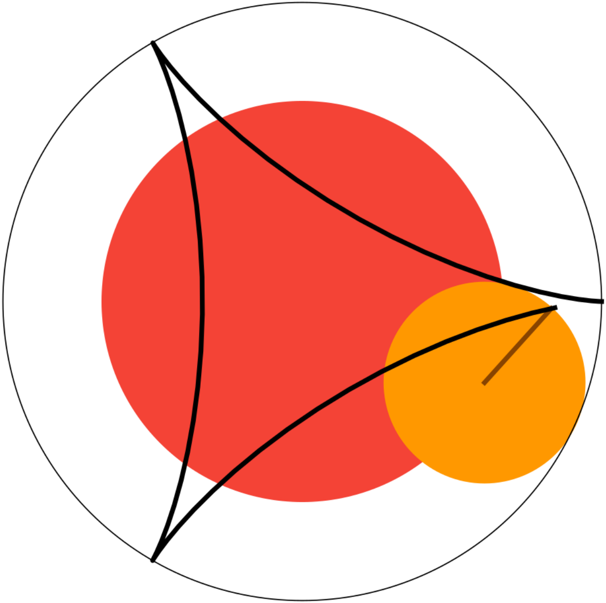

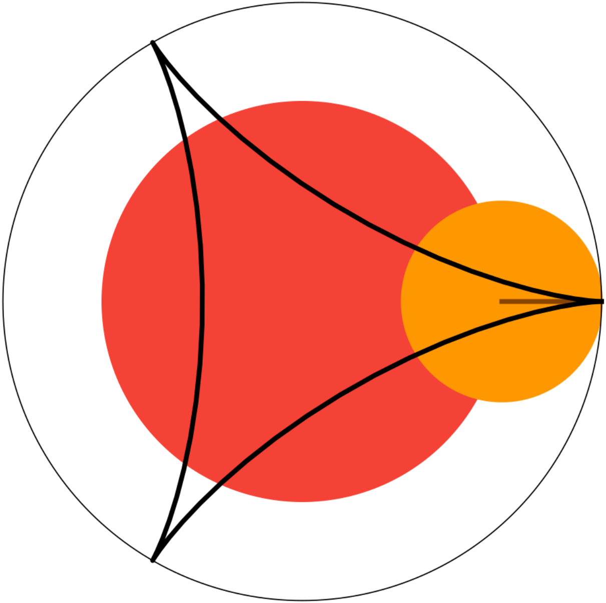

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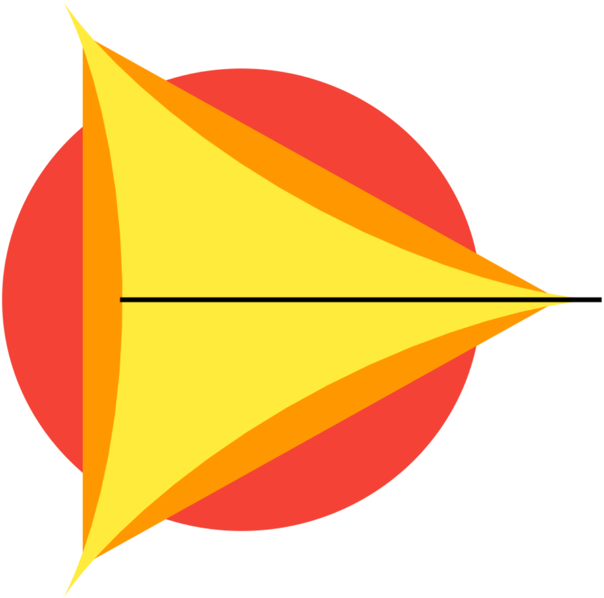

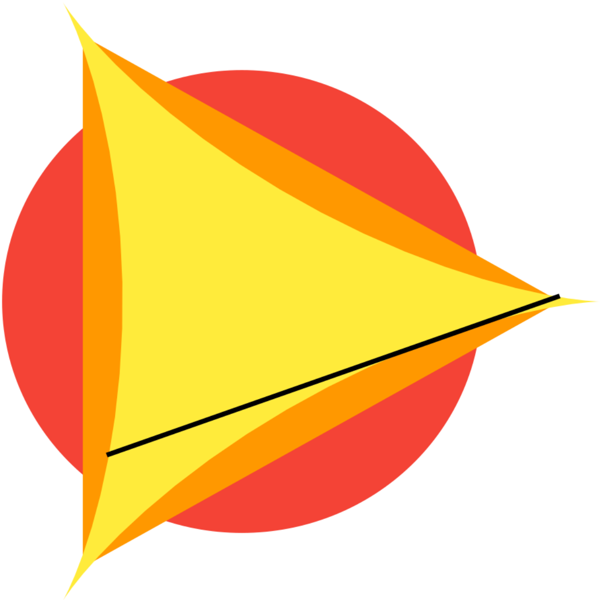

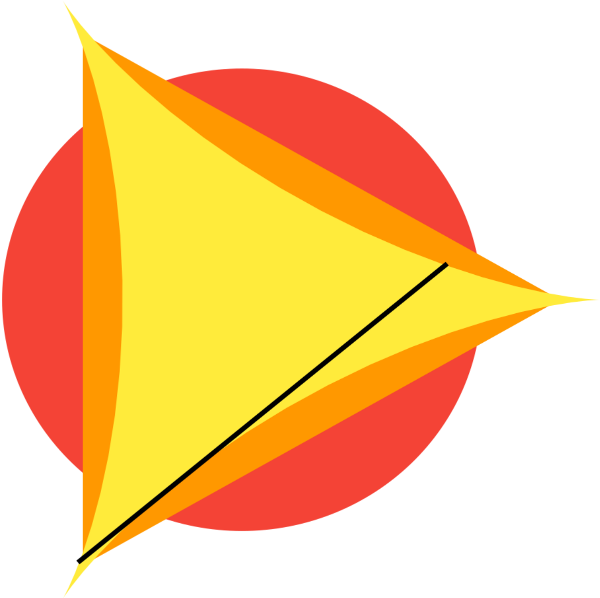

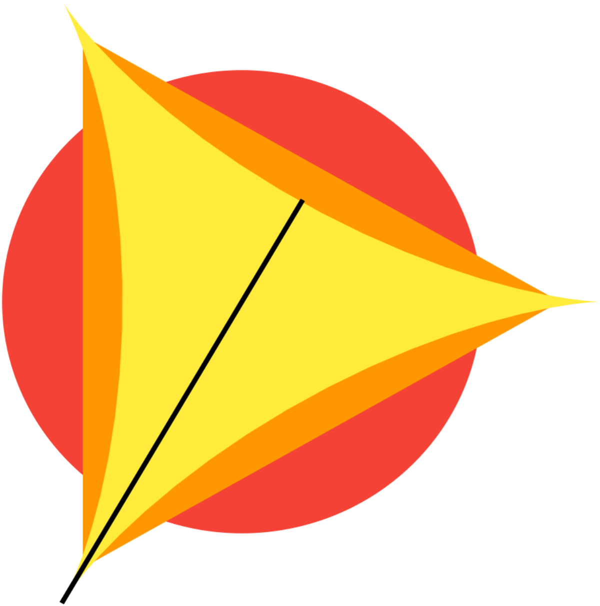

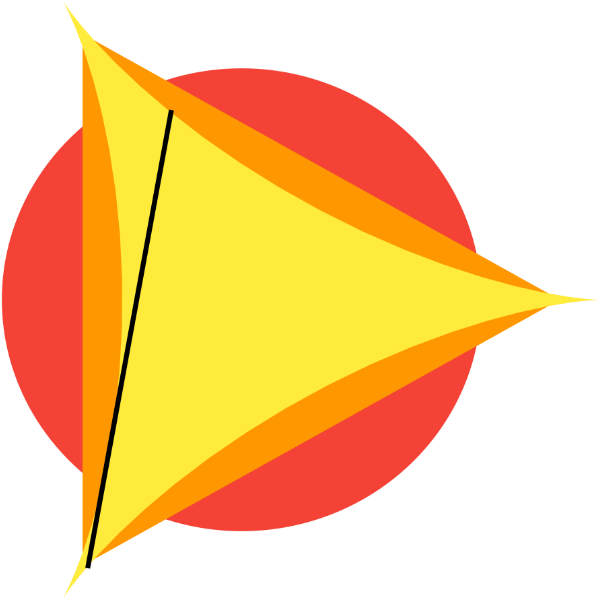

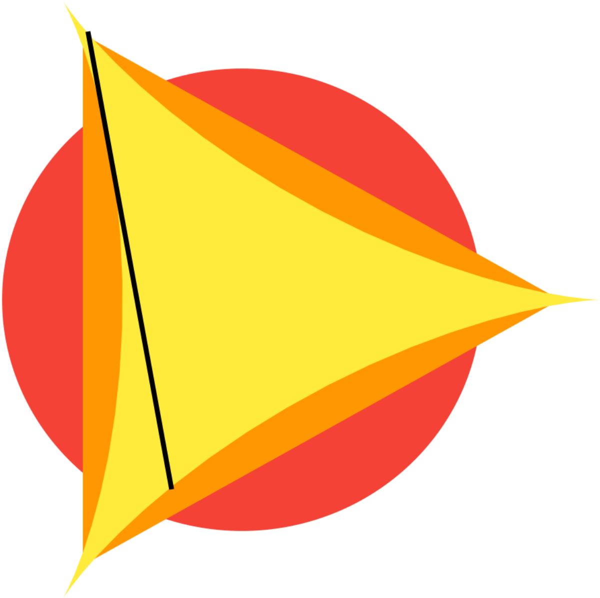

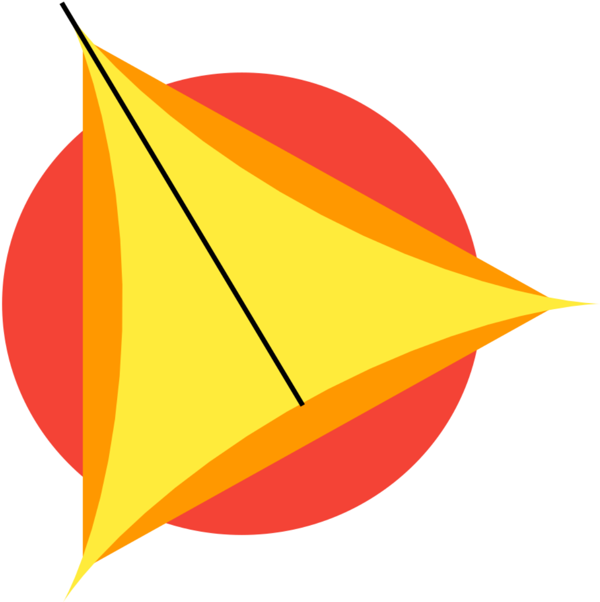

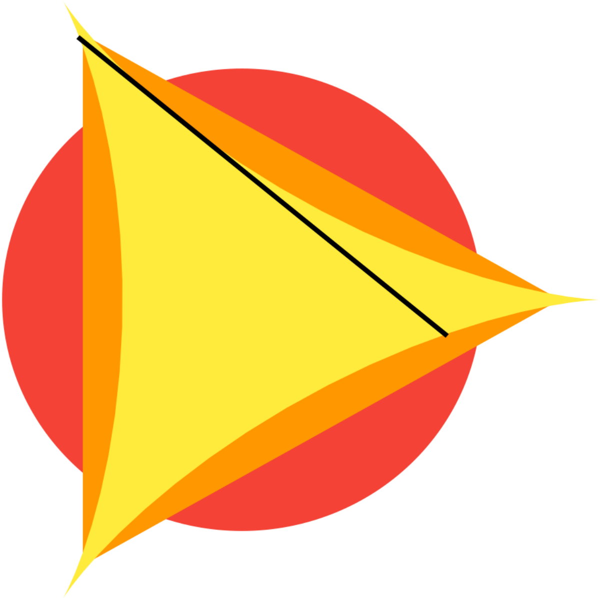

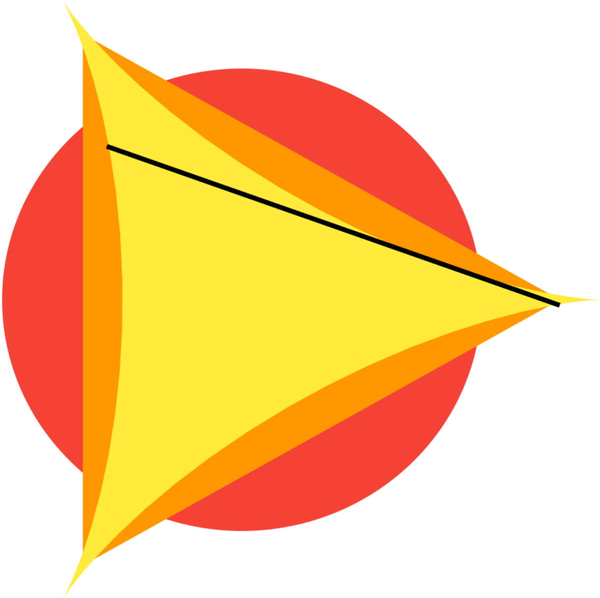

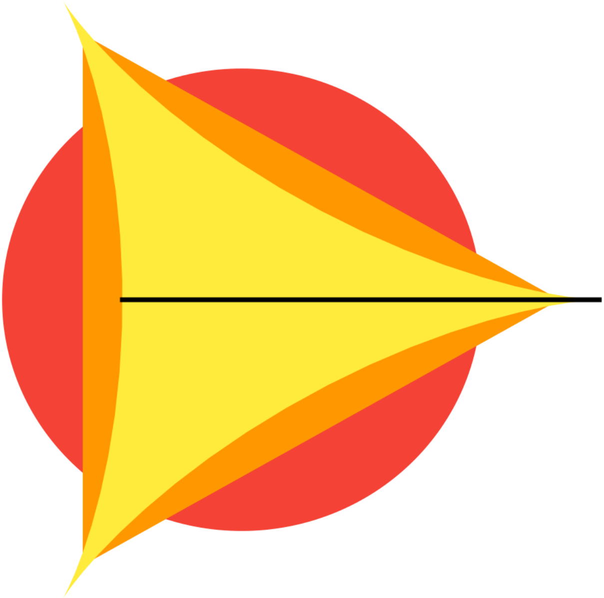

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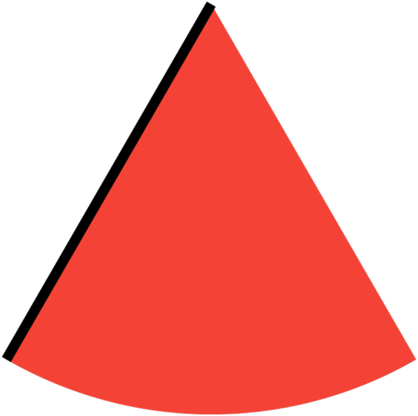

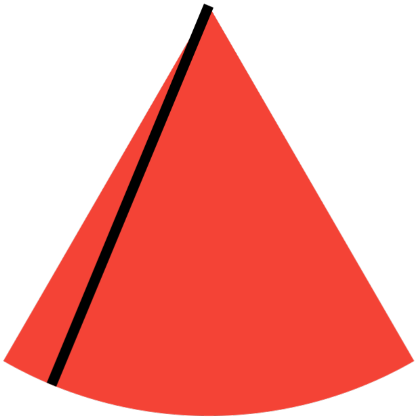

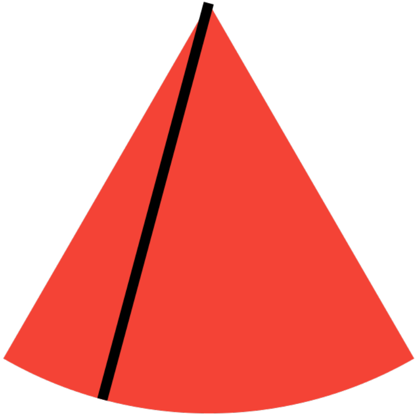

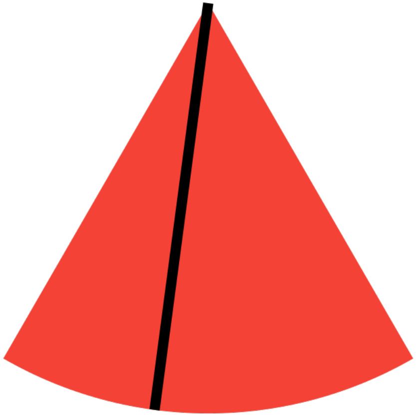

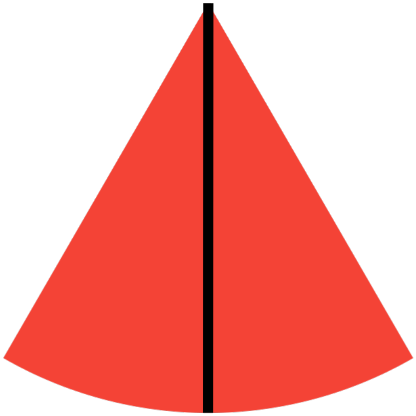

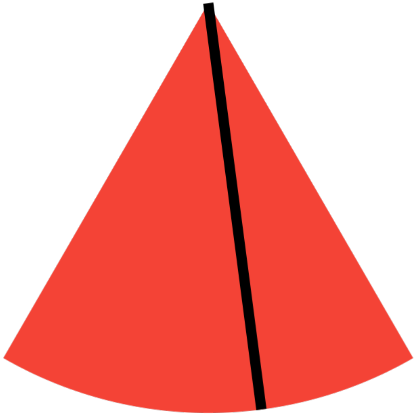

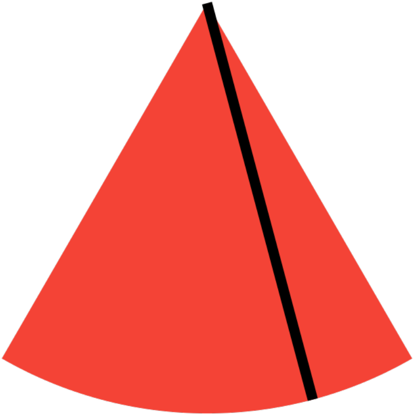

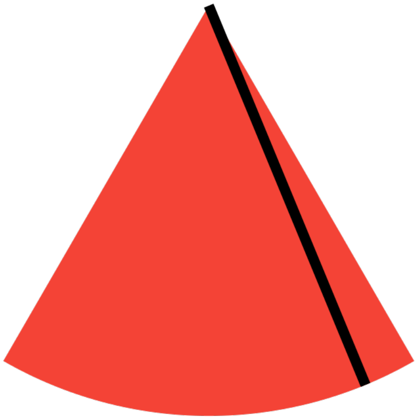

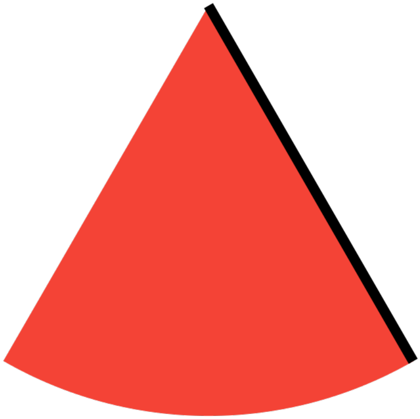

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- So if the area grows by a roughly constant amount at each step, then the final rescaled thing will have area like $(N + 1)/(N + 1)^2 = 1/(N + 1) \to 0$.

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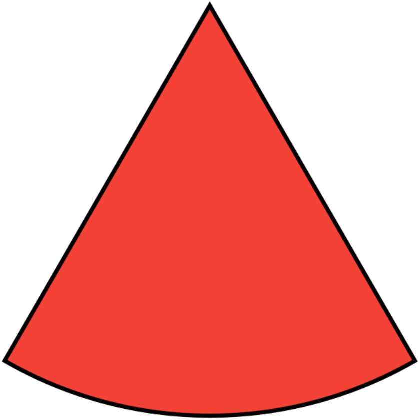
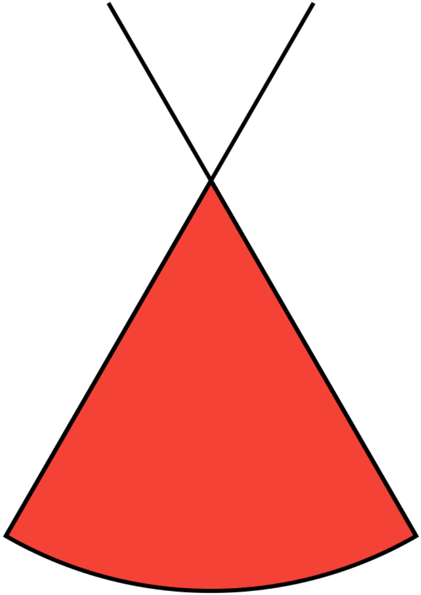
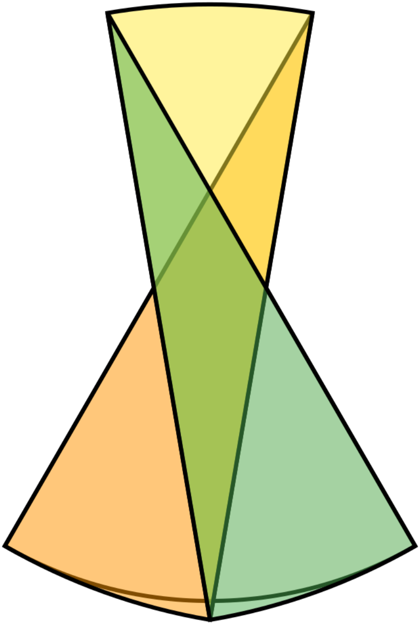
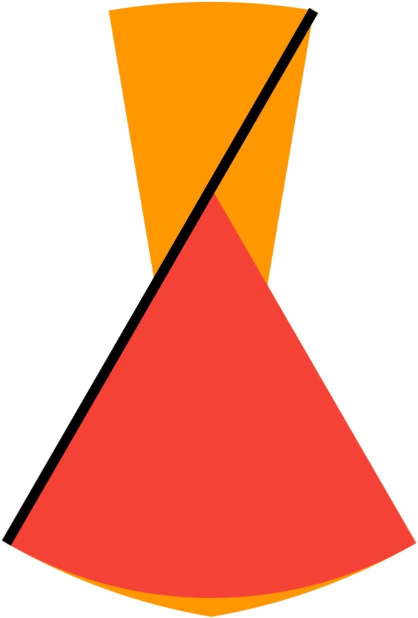
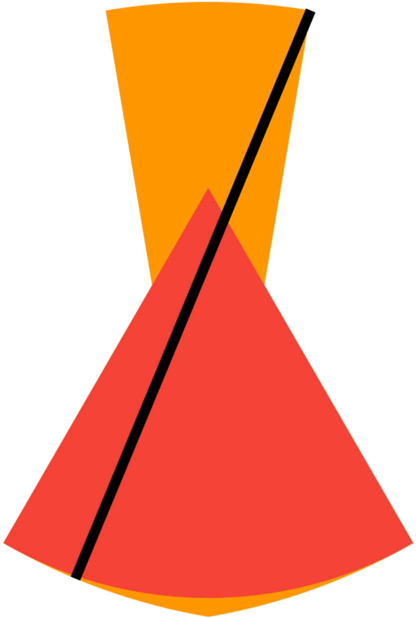
- Another key point is that if two sweeps align along an edge, then after the iteration, they will still align except possibly for the need of a shift.

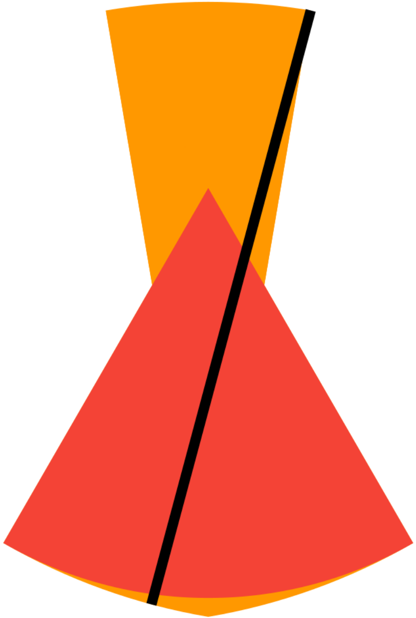


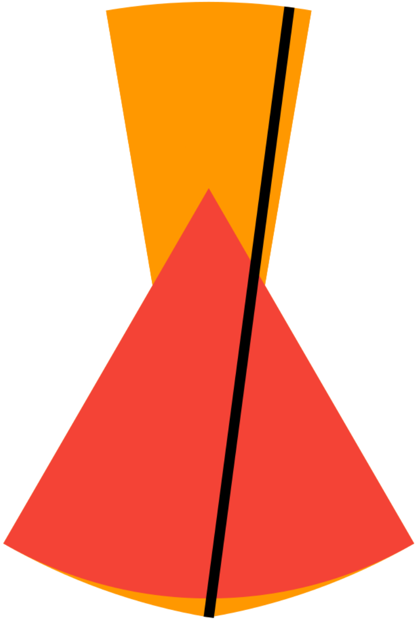


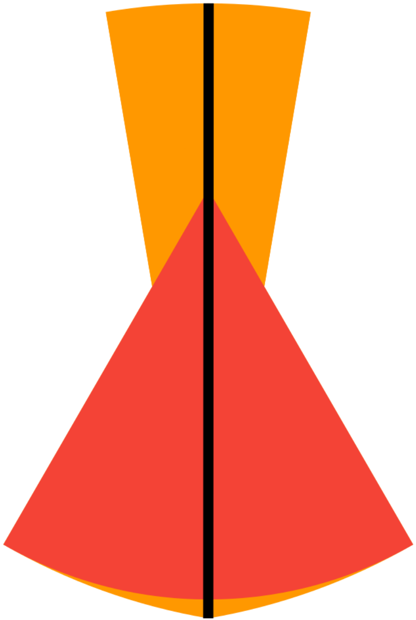


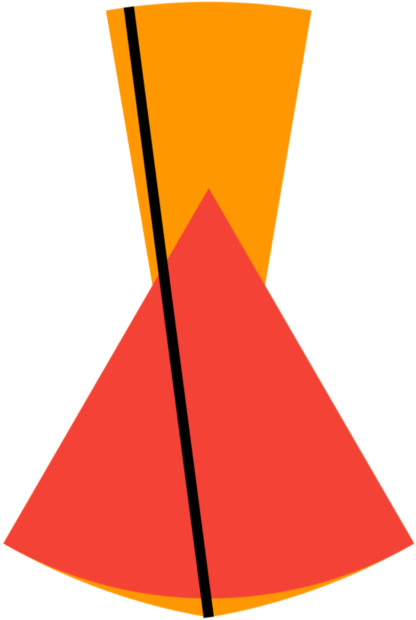


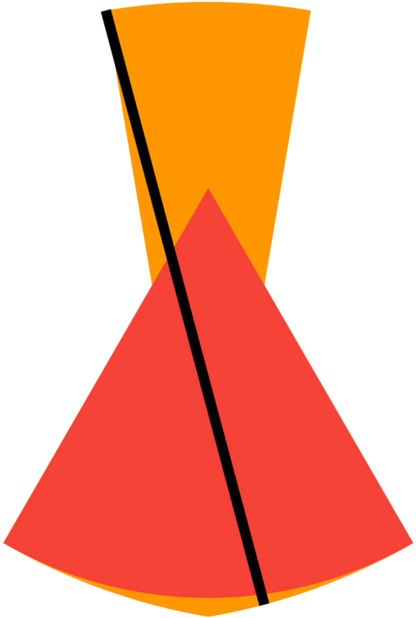


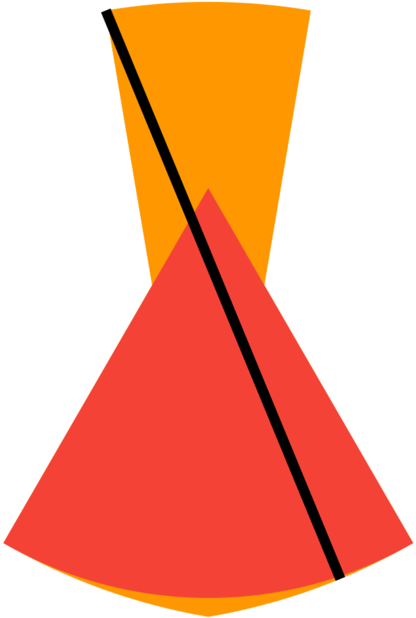


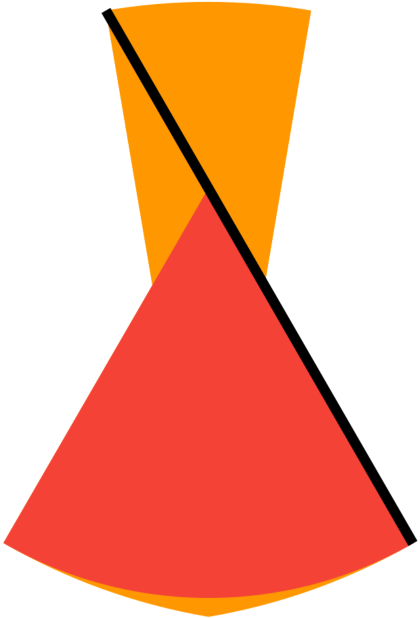






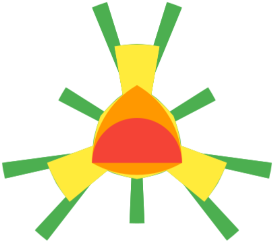


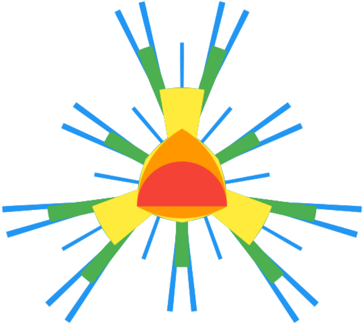


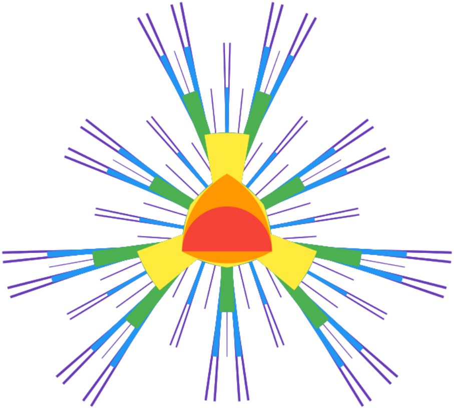


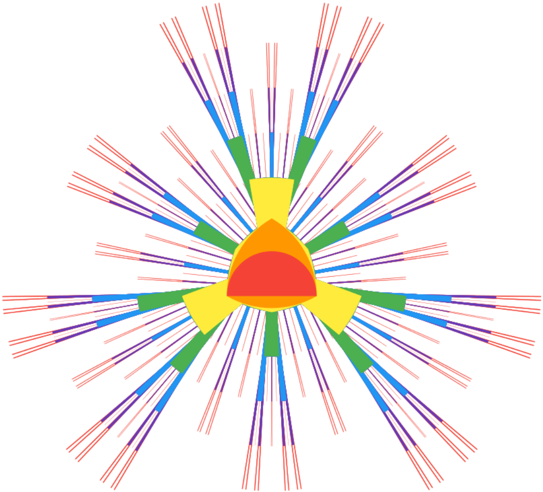
21. Carrying out the Process

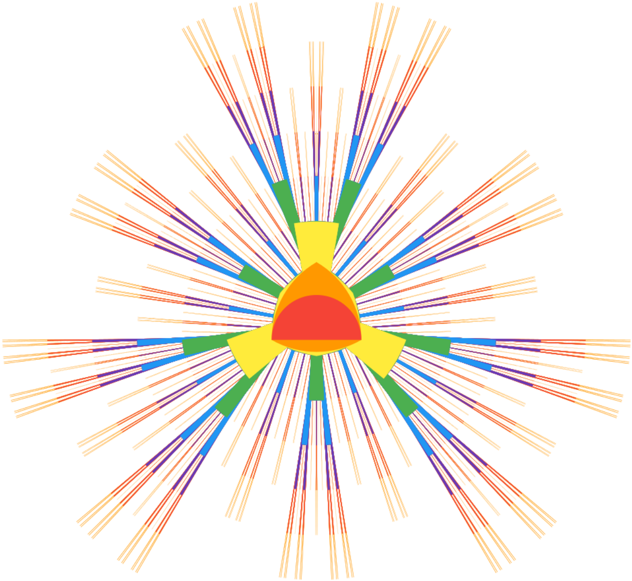
Let's see what this iteration process gives us when we start with a single 180 degree sweep followed by a slide...



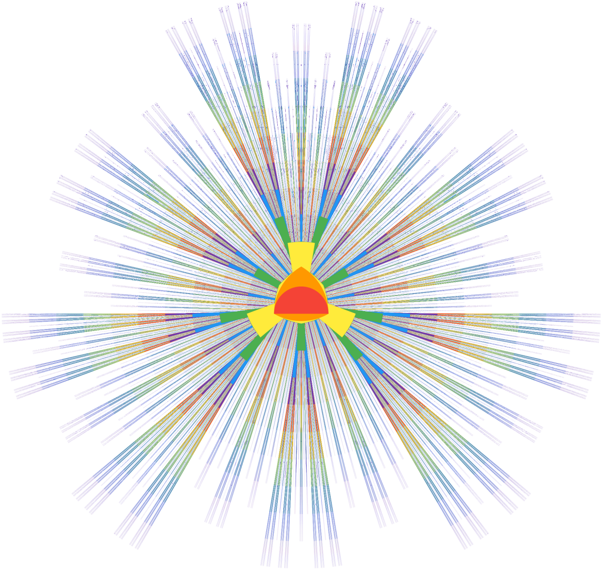








23. Twelve Iterations



https://www.youtube.com/watch?v=pWk57HpPJmQ

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An Easier Question

We will try to answer a simpler question: A set with large area must also have large _______.

Sets with Large Area Must Have Large Diameter

Isodiametric Inequality

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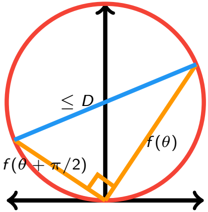

Proof (from Littlewood's miscellany, p. 32). Suppose that the region sits on top of the x-axis and is given by the graph of $0 \le r \le f(\theta)$ for $0 \le \theta \le \pi$. We use polar coordinates to compute area and do a clever manipulation to find a right triangle:

$$A = \frac{1}{2}\int_0^{\pi} (f(\theta))^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(f(\theta))^2 + \left(f\left(\theta + \frac{\pi}{2}\right)\right)^2\right] d\theta$$
$$\le \frac{1}{2}\int_0^{\frac{\pi}{2}} D^2 = \frac{\pi D^2}{4}.$$

Note equality holds for all disks.

26. Measures and Nonconcentration Inequalities

- A **measure** μ is a generalization of area which allows for other ways quantifying size of sets E. The key feature is that the measure of a disjoint union of sets is the sum of the measures (e.g., the area of two non-overlapping disks is the sum of the areas of the individual disks).

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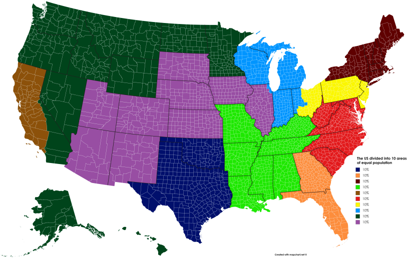

https://www.reddit.com/r/MapPorn/comments/9xg11l/oc_the_us_divided_into_10_areas_of_equal/

Two More Facts About Area and Diameter

Theorem: Isodiametric Inequality Rewritten

If E is a nice planar region of area $\text{area}(E)$, then it is always possible to find two points a, b in E such that

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Theorem: Area Extremizes the Isodiametric Inequality

Suppose μ is any measure of planar regions. If

$$\mu(E) \leq \frac{\pi}{4}(\operatorname{diam}(E))^2$$

for all regions E, then $\mu(E) \leq \operatorname{area}(E)$.

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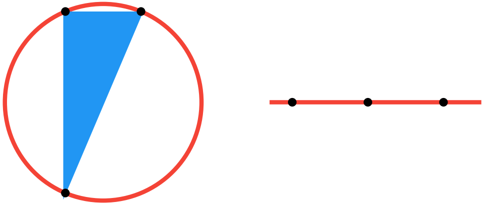

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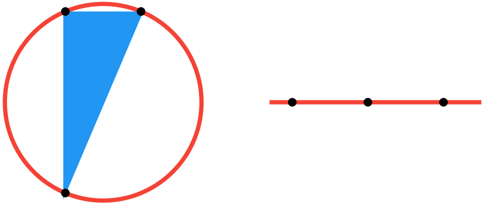

- Sets E in the unit circle satisfy an inequality of the form

$$\text{max triangle size}(E) \geq c(\text{arc length } E)^3$$

while sets inside a line segment satisfy no such inequality.

29. A Host of Related Questions

- This is the entryway of a deep rabbit hole: For example, sometimes there are things you'd like to know about vectors, matrices, polynomials, or other objects instead of points:
  - When can you use largeness of matrix entries to determine largeness of the determinant?

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- These questions may seem toy-ish or artificial, but they have deep implications for "serious" mathematical questions.
- These are all cases in which something concrete can be said, and there is more interesting mathematics out there about which we currently understand little.

Thank You For Your Attention!