How Big Can it Be? Some Challenges of Size in Fourier Analysis

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The main part of the talk will be about the Kakeya Needle Problem, which examines whether sets which are large enough to move a needle-shaped object around in must also be large in the usual sense of area. This problem has an interesting and satisfying solution, but is also intimately connected to a host of open questions, large and small, in harmonic analysis. As time permits, we will explore connections to geometric nonconcentration inequalities, which are a general framework for figuring out how to define largeness of sets so that it corresponds with whatever geometric properties that you find interesting.
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The Kakeya Needle Problem: Formulation
Consider all regions \( U \) in the plane such that:
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4. A First Idea: Circles

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Let’s see what this looks like...
Can we beat the circle?

In any optimization question like this one, once a candidate is identified, the question is whether there is a better one. For us, the question is:

- Among convex sets, is there a smaller set than the circle of radius $\frac{1}{2}$ in which a unit line segment can be rotated through 180 degrees?

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Connections to Kakeya

**The Kakeya Conjecture.** We define a Besicovitch set in $\mathbb{R}^n$ to be a set which contains a unit line segment in every direction. Such sets can have arbitrarily small (even zero) Lebesgue measure. Does the set have $n$-dimensional fractal measure ($n > 2$)?
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**PDEs.** The Kakeya conjecture is connected to regularity properties of PDEs. In particular, certain conjectured estimates on the regularity of solutions of the wave equation would imply Kakeya.
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11. Non-convex Regions: Deltoids

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Let’s see what that looks like.
A Geometric Property of Deltoids

A Very Close Fit

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- Let’s see what a sweep looks like...
17. A Change of Perspective

- It is slightly easier to think about the iteration process in the following way.

- We will start with a good region built from only slides and sweeps which works for a needle of some length $N$.

- We will adjust the region into new slides and sweeps. Rather than making the region smaller, we will make it bigger, but we will also insist that the bigger region can accommodate longer needles, e.g., needles of length $N + 1$.

- Then the problem is about competing rates of growth: if the region accommodates a needle of length $N + 1$, then we could shrink the whole thing down by a factor of $N + 1$ in each direction. This reduces the area by a factor of $(N + 1)^2 - 2$.

- So if the area grows by a roughly constant amount at each step, then the final rescaled thing will have area like $(N + 1)/(N + 1)^2 = 1/(N + 1) \to 0.$
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- A key point is that in replacing the old sweep, the starting angle and ending angle of the needle do not change.
- Another key point is that if two sweeps align along an edge, then after the iteration, they will still align except possibly for the need of a shift.
21. Carrying out the Process

Let's see what this iteration process gives us when we start with a single 180 degree sweep followed by a slide...
23. Twelve Iterations

https://www.youtube.com/watch?v=pWk57HpPJmQ
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Kakeya Conjecture (Hard)

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The conjecture is known only when $n = 2$. 
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An Easier Question
We will try to answer a simpler question: A set with large area must also have large ________.
Sets with Large Area Must Have Large Diameter

Isodiametric Inequality
If $A$ is a planar region with diameter $D$, then

$$A \leq \frac{\pi D^2}{4}.$$
Sets with Large Area Must Have Large Diameter

Isodiametric Inequality

If \( A \) is a planar region with diameter \( D \), then

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Proof (from Littlewood’s miscellany, p. 32). Suppose that the region sits on top of the \( x \)-axis and is given by the graph of \( 0 \leq r \leq f(\theta) \) for \( 0 \leq \theta \leq \pi \). We use polar coordinates to compute area and do a clever manipulation to find a right triangle:

\[
A = \frac{1}{2} \int_{0}^{\pi} (f(\theta))^2 \, d\theta = \frac{1}{2} \int_{0}^{\pi/2} \left[ (f(\theta))^2 + (f(\theta + \pi/2))^2 \right] \, d\theta
\]

\[
\leq \frac{1}{2} \int_{0}^{\pi/2} D^2 = \frac{\pi D^2}{4}.
\]

Note equality holds for all disks.
26. Measures and Nonconcentration Inequalities

- A **measure** \( \mu \) is a generalization of area which allows for other ways quantifying size of sets \( E \). The key feature is that the measure of a disjoint union of sets is the sum of the measures (e.g., the area of two non-overlapping disks is the sum of the areas of the individual disks).

![Image](https://www.reddit.com/r/MapPorn/comments/9xg11l/oc_the_us_divided_into_10_areas_of_equal/)
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A common example is the population measure: If $E$ is a region on the surface of the globe, then $\mu_{pop}(E)$ denotes the number of people living in region $E$. 
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- A common example is the population measure: If \( E \) is a region on the surface of the globe, then \( \mu_{\text{pop}}(E) \) denotes the number of people living in region \( E \).

- Population size does not correspond with geographic size.

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Two More Facts About Area and Diameter

Theorem: Isodiametric Inequality Rewritten

If $E$ is a nice planar region of area $\text{area}(E)$, then it is always possible to find two points $a, b$ in $E$ such that

$$\text{dist}(a, b) \geq \sqrt{\frac{4 \text{area}(E)}{\pi}}.$$
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**Theorem: Area Extremizes the Isodiametric Inequality**
Suppose $\mu$ is any measure of planar regions. If

$$\mu(E) \leq \frac{\pi}{4} (\text{diam}(E))^2$$

for all regions $E$, then $\mu(E) \leq \text{area}(E)$. 
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In a given set $E$ in the plane, when can we find three points $a, b, c$ in the set $E$ which when joined together form a triangle of large area? Such sets do not need to have positive areas, but they cannot be flat:

Sets $E$ in the unit circle satisfy an inequality of the form

$$\max \text{ triangle size}(E) \geq c(\text{arc length } E)^3$$

while sets inside a line segment satisfy no such inequality.
29. A Host of Related Questions

- This is the entryway of a deep rabbit hole: For example, sometimes there are things you’d like to know about vectors, matrices, polynomials, or other objects instead of points:
  - When can you use largeness of matrix entries to determine largeness of the determinant?
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  - What if I have a large batch of different matrices. Is there necessarily one with a large determinant?
  - What notions of largeness in a vector space guarantee that large sets always have “very linearly independent” bases associated to them?

These questions may seem toy-ish or artificial, but they have deep implications for “serious” mathematical questions. These are all cases in which something concrete can be said, and there is more interesting mathematics out there about which we currently understand little.
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- This is the entryway of a deep rabbit hole: For example, sometimes there are things you’d like to know about vectors, matrices, polynomials, or other objects instead of points:
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  - What if I have a large batch of different matrices. Is there necessarily one with a large determinant?
  - What notions of largeness in a vector space guarantee that large sets always have “very linearly independent” bases associated to them?
  - What notions of largeness in the plane guarantee that large sets always have points which are far from lying on algebraic curves (i.e., three points far from a line, four points far from a circle, six points far from a conic section,...)
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Thank You For Your Attention!