1+2 FTC for $\int_C$

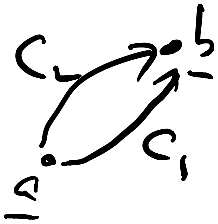
$\Downarrow$

$F = \nabla\varphi \Longleftrightarrow \int_C F\cdot d\alpha$ indep of path

$\Updownarrow$

2nd $\Longrightarrow$

1st $\Longleftarrow$



$\oint_{C_0} F\cdot d\alpha = 0$ ← loop

C_0: C_1 then C_2 reverse

$\oint \quad \oint$

Conservative

F is a gradient

→ $F = \nabla\varphi \Longrightarrow D_i f_j = D_j f_i$

$(f_1, \dots, f_n)$ ↑ potential fn

$D_i D_j \varphi = D_j D_i \varphi$

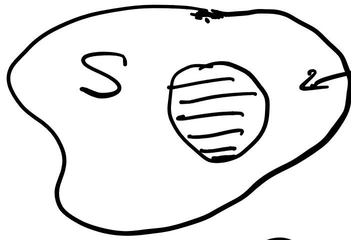
Ex 2; p 341 $\qquad S = \mathbb{R}^2 - 0$

$$F = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

Converse fails

→ 

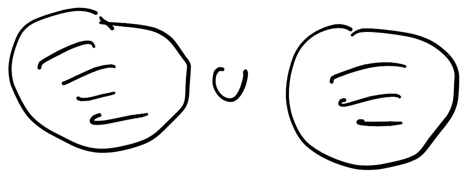
$\oint_C F\cdot d\alpha = 2\pi \neq 0$



F

converse holes

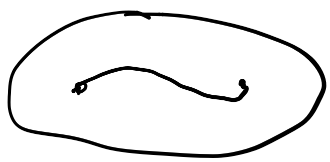
S: simply conn.



not conn

dis conn

disjoint $\bigcup$ of open sets



path conn.

$\Updownarrow$ open

conn

V.f. $\leftrightarrow$ diff forms

$\mathbb{R}^2$ $\qquad F = (f_1, f_2) = f_1\vec{i} + f_2\vec{j}$

$$\updownarrow$$

diff form $\quad \omega = f_1\,dx + f_2\,dy$

C curve, param by

$$\alpha : [a, b] \longrightarrow \mathbb{R}^2$$

$$\alpha(a) = \underline{a}, \quad \alpha(b) = \underline{b}$$

$$\alpha = (\alpha_1, \alpha_2) \qquad \alpha_1(t) = x$$

$\alpha_2(t) = y$

Say $F = \nabla\varphi = \left(\dfrac{\partial\varphi}{\partial x}, \dfrac{\partial\varphi}{\partial y}\right)$

$$F = (f_1, f_2)$$

$$\begin{aligned}\omega &= f_1\,dx + f_2\,dy \\ &= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy \\ &= d\varphi\end{aligned}$$

2D FTC: $\nabla\varphi$

$$\int_C F\cdot d\alpha = \varphi(\underline{b}) - \varphi(\underline{a}) = \varphi\Big|_{\underline{a}}^{\underline{b}}$$

$$\| $$

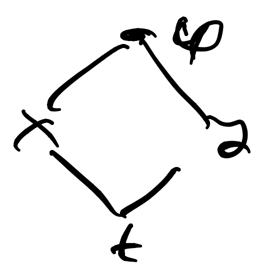
$$\int_a^b f_1\frac{dx}{dt}dt + \int_a^b f_2\frac{dy}{dt}dt$$

$$\| $$

$$\int_a^b\left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt}\right)dt$$

$$\| $$

$$\int_a^b \frac{d\varphi}{dt}dt = \int_C d\varphi$$



Ex 2 p 341

on $S = \mathbb{R}^2 - 0$

vf F on S

$\updownarrow$

diff. form ω on S

$\omega = d\theta$ not d(functn)

(r, θ) $\quad r \geq 0$ $\quad F \neq \nabla$(fn)

Not Simply Conn

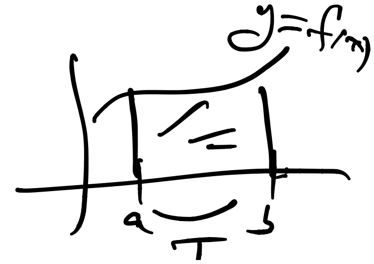
$d\theta$ well def away from 0.

More – in Chap 11
– re Green's Thm.

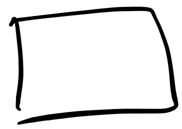
Chap 11 Multiple $\int$'s.

Idea:

$$\int_a^b f(x)\,dx$$



$f(x,y)$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}^+$

S  $\subset \mathbb{R}^2$

rectangle

Graph of $z = f(x,y)$.

Volume under, over S.

$\rightarrow \displaystyle\iint_S f(x,y)\, dA$

area

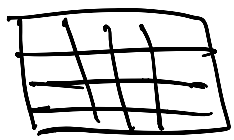
$dx\,dy$

Step fns, lower $\int$, upper $\int$

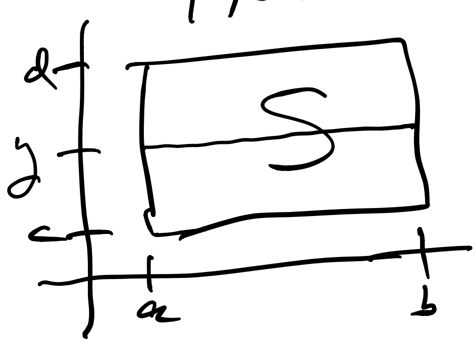
If $=$: integrable.

Same here. $\qquad$ Th 11.5, p. 358

Fubini's Thm



How to compute: $\qquad$ Say f int'ble on S.



$\displaystyle\iint_S f(x,y)\, dA$

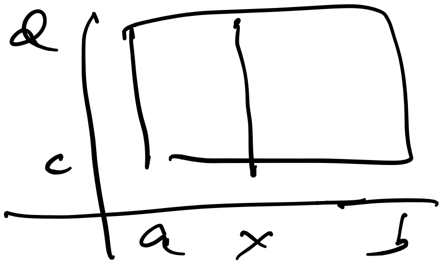
$\parallel$

$\rightarrow \displaystyle\int_{y=c}^{d} \underbrace{\left(\int_{x=a}^{b} f(x,y)\, dx\right)}_{\text{constant}} dy$

Pf:

Check for step fn. $\qquad$ $\parallel$ cross-

Sup, infs. $\qquad \| \qquad$ Sectional area



$$\to \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx$$

§ 11.8 $\qquad$ worked out ex's.

---

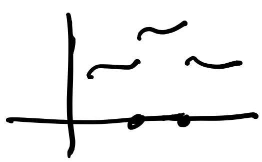
f cont $\Rightarrow$ f int

Cont on closed bounded region $\Longrightarrow$ bounded

Small span thm $\leftrightarrow$ unif. cont.

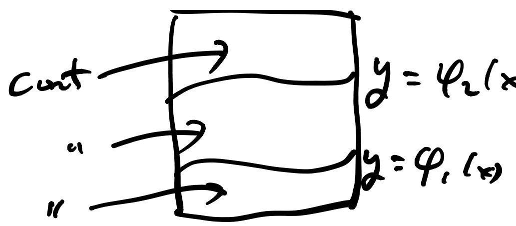
Same pf $\quad$ pp 363-364

Piecewise ~~cont~~:



1 vbl: p.c. $\Rightarrow$ int.

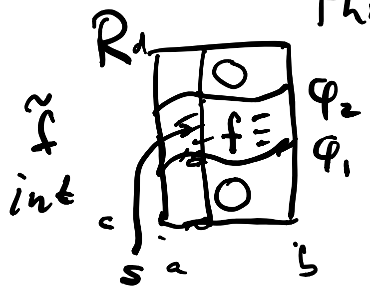
f(x,y) p.c.: cont exc. on fin. many curves (graph of cont fn)



$\Rightarrow$ int.

pf as before via sm. sp. thm.

Thm. 11.7, 11.8



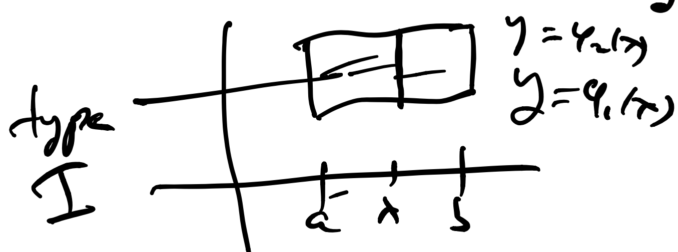
Def: $\iint_S f\, dA = \iint_R \tilde{f}\, dA$

$$\iint_S f\, dA = \iint_R \tilde{f}\, dA = \int_{x=a}^{b} \Big(\underbrace{\int_{y=c}^{d} \tilde{f}(x,y)\, dy}_{} \Big) dx$$

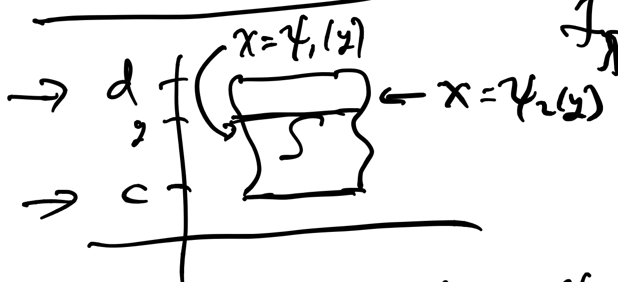
$$= \int_{x=a}^{a} \Big(\int_{y=\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \Big) dx$$

$$\int_{y=\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x,y)\, dy = \int_{y=\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy$$

type I

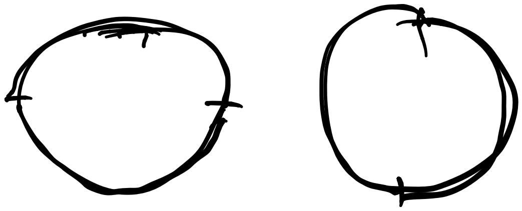


Type II

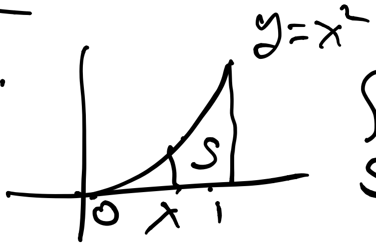


$$\iint f\, dA = \int^{d} \Big(\int^{\psi_2(y)} f(x,y)\, dx \Big) dy$$

S $\qquad y = c \quad x = \varphi_1(y)$



Ex.



$$\iint_S xy\,dA$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{x^2} xy\,dy\right)dx$$

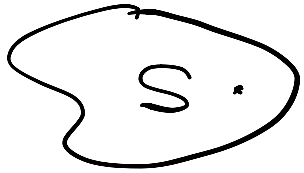
$$\frac{xy^2}{2}\Big|_{y=0}^{x^2} = \frac{x^5}{2}$$

$$= \int_{x=0}^{1} \frac{x^5}{2}\,dx$$

$$= \frac{x^6}{12}\Big|_0^1 = \frac{1}{12}$$

§10.14 – examples

§10.16 appl. to physics



$f(x,y) =$ density at (x,y)

$$\text{Mass} = \iint_S f(x,y)\,dA$$

$a \qquad b$

$g(x)$

$\int g(x)\,dx$

If Density 1:

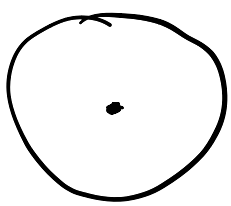
$$\text{mass} = \iint_S 1\,dA = \iint_S dA = \text{area of } S$$

If density varies

$$\text{avg density} = \frac{\text{mass}}{\text{area}} \qquad \text{mass} \leftarrow \iint_S f(x,y)\,dA \qquad \text{area} \leftarrow \iint_S dA$$

Center of mass

$$(\bar{x}, \bar{y}) \qquad \bar{x} = \frac{\iint_S x\, f(x,y)\,dA}{\iint f(x,y)\,dA}$$



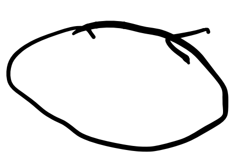
If density is const

– centroid

Green's Thm

$$\varphi : \overset{\text{open}}{S} \longrightarrow \mathbb{R}$$

Cont diff $\quad S \subset \mathbb{R}^2$

C  in S

$$\alpha : [a,b] \longrightarrow S$$

$$\oint_C \nabla\varphi \cdot d\alpha = 0$$

$$\nabla \varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j}$$

$$\oint_C d\varphi \qquad d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

$$P = \frac{\partial \varphi}{\partial x} \qquad Q = \frac{\partial \varphi}{\partial y}$$

$$F = \nabla \varphi = P\vec{i} + Q\vec{j}$$

$$d\varphi = P\,dx + Q\,dy$$

$$\frac{\partial}{\partial y}\frac{\partial \varphi}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y}$$

Converse? If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$,

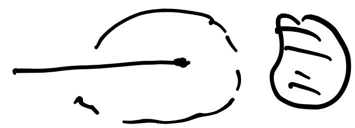
& $F = P\vec{i} + Q\vec{j}$, is F a gradient?

Not always

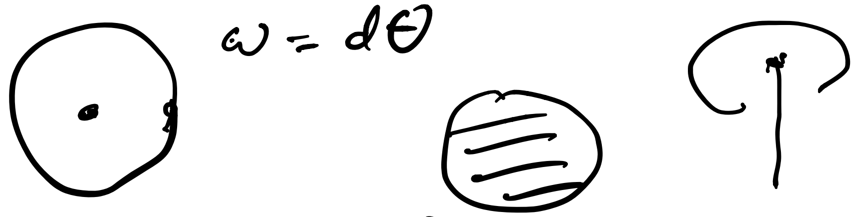
Ex 2 p. 341

not simp. conn

$S = \mathbb{R}^2 - O$



$\omega = d\theta$



Converse OK for S.C. regions?

P, Q on S

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

then $\oint_C P\,dx + Q\,dy = 0$

$\Updownarrow$

$F = P\vec{i} + Q\vec{j}$ is a gradient

Reason: Green's Thm

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

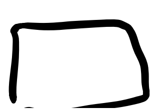
$R^{\subseteq S}$ enclosed by C

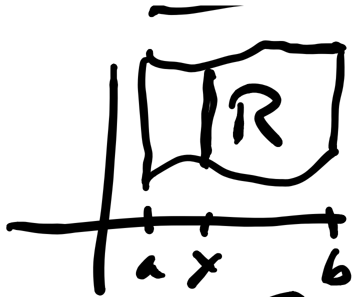
So if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then RHS $= 0$

$\therefore$ LHS $= 0$.

$$\iint_R \cdots\, dA = \int_{y=c}^{d} \left(\int_{x=a}^{b} \cdots\, dx \right) dy$$

$dx\,dy$



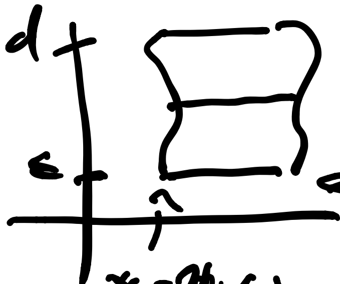


$y = \varphi_2(x)$

$y = \varphi_1(x)$

$$\int_{x=a}^{b}\left(\int_{y=\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)dx$$

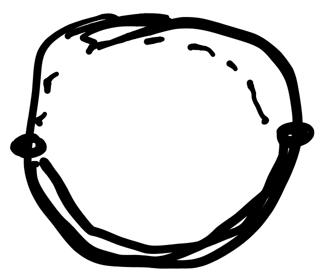
Type I

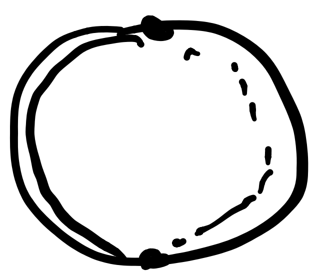


$$\int_{\cdots}^{\cdots}\left(\int_{\cdots}^{\cdots} \cdots \, dx\right)dy$$

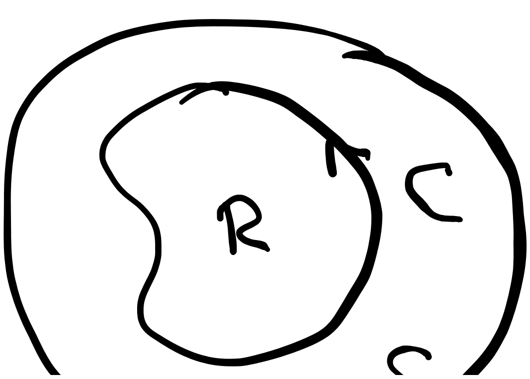
$x = \psi(y)$ Type II

Both types

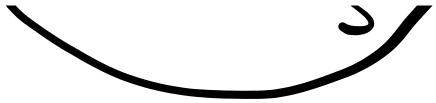




Green's Thm



P, Q



$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Ex. C : $x^2+y^2=1$

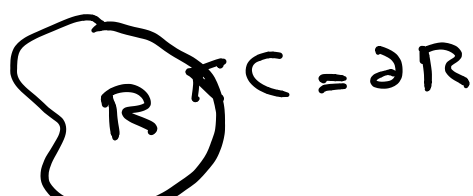
R : $x^2+y^2 \le 1$

$$\oint_C \underbrace{(2y + e^{x^2})}_{P}\,dx + \underbrace{(3x - \cos(e^{y}))}_{Q}\,dy$$

$$= \iint_R (3-2)\,dA = \iint_R dA = \text{area}(R)$$

$$= \pi$$

Ex.



$$\text{area}(R) = \iint_R dA$$

$$= \iint 1\,dA$$

$$1 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad R$$

Ex. 1) $Q = x,\ P = 0$

2) $P = -y,\ Q = 0$

3) $P = -\frac{1}{2}y,\ Q = \frac{1}{2}x$

$\vdots$

$$\text{area}(R) = \oint_C x\,dy = -\oint_C y\,dx$$

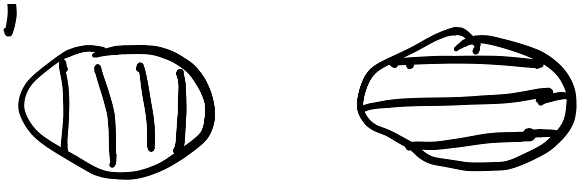
$$= \frac{1}{2}\left(\oint_C x\,dy - \oint_C y\,dx\right)$$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Pf of Green's thm:

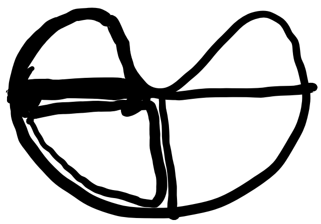
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Special case: R: I and II



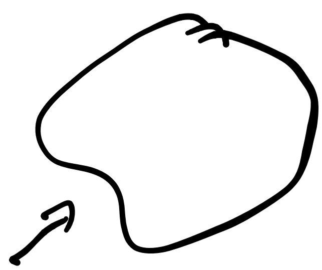
$-\iint_R \frac{\partial P}{\partial y}\, dA$, eval using I,
get $\oint_C P\, dx$.

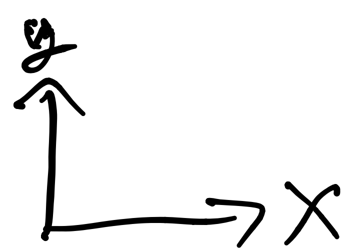
$\iint \frac{\partial Q}{\partial x}\, dA$, eval using II,
get $\oint_C Q\, dy$



$\iint_R = \oint_C$
on each piece
$\Rightarrow$ OK on whole region

p 382



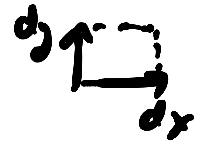


$\iint \cdots dA$
? $dx\, dy$

$\omega = P\,dx + Q\,dy$ diff 1-form

φ " 0-form

$f\,\underbrace{dx \wedge dy}_{\text{oriented area elt.}}$



$$dy \wedge dx = -dx \wedge dy$$

$$\left(\int_b^a \cdots = -\int_a^b \cdots \right)$$

$$\omega = P\,dx + Q\,dy$$

$$\to d\omega = \underline{dP} \wedge dx + \underline{dQ \wedge dy}$$

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy \leftarrow$$

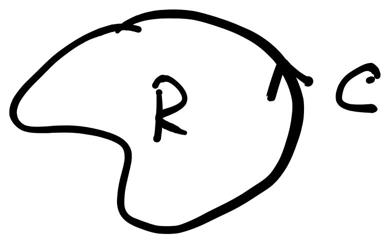
$$d\omega = \frac{\partial P}{\partial y}\underline{dy \wedge dx} + \frac{\partial Q}{\partial x}\underline{dx \wedge dy}$$

$$= \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\underline{dx \wedge dy}$$

$dx \wedge dx = 0 \longrightarrow$

$dy \wedge dy = 0$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$



$\omega = P\,dx + Q\,dy$

$$\oint_{\partial R} \omega = \iint_R d\omega$$

Green's Thm

∂R

2^P FTC

$\alpha : [a, b] \to C$

$\alpha(a) = \underline{a}$

$\alpha(b) = \underline{b}$

$\varphi : C \xrightarrow[\text{diff}]{} \mathbb{R}$

$$\int_C d\varphi = \varphi\Big|_{\underline{a}}^{\underline{b}} = \varphi(\underline{b}) - \varphi(\underline{b})$$

0-dim $\int$

C

∂C

$\underline{a}$	$\underline{b}$
$-$	$+$