Change of vbls §11.26

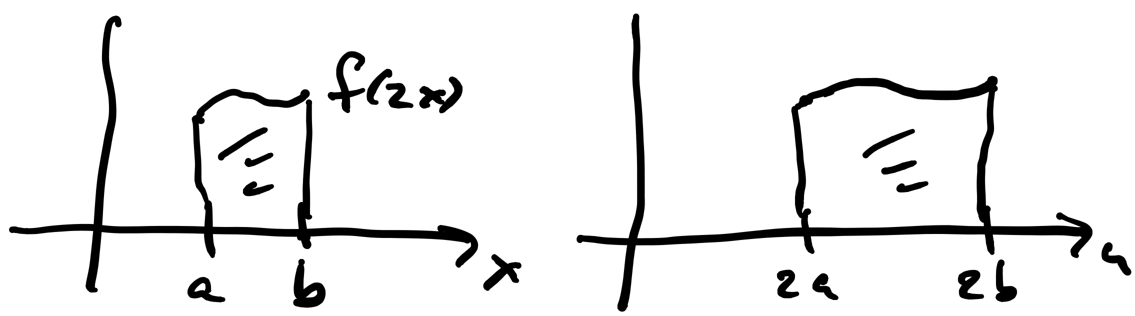
1 vbl case

$u = g(x)$

$$\int_{x=a}^{b} f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du \,=\, \frac{du}{dx}dx} = \int_{u=g(a)}^{g(b)} f(u)\,du$$

Ex. $u = 2x$ $\quad du = 2dx$

$$\to \quad 2\int_{x=a}^{b} f(2x)\,dx = \int_{u=2a}^{2b} f(u)\,du$$



2 vbl case

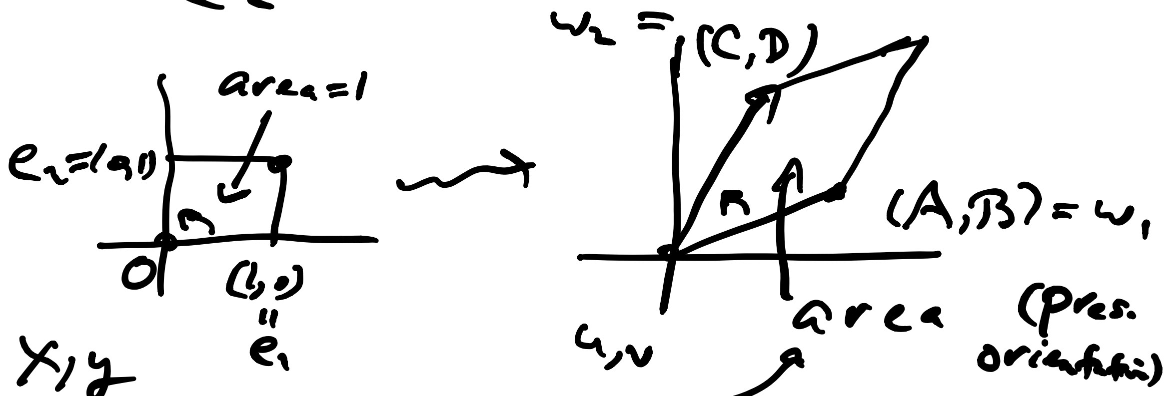
fn on $\mathbb{R}^2$

x, y

Ex. $(x,y) \rightsquigarrow (u,v)$

$$u = Ax + Cy = U(x,y)$$
$$v = Bx + Dy = V(x,y)$$

$$(0,0) \rightsquigarrow (0,0)$$
$$e_1 = (1,0) \rightsquigarrow (A,B)$$
$$e_2 = (0,1) \rightsquigarrow (C,D)$$



area (pres. orientation)

$$= \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC = \Delta > 0$$
$$= \|w_1 \times w_2\|$$

$$\iint_R f(U(x,y), V(x,y))\, \Delta \underbrace{dA}_{dx\,dy}$$

$$= \iint_{R^*} f(u,v)\, \underset{\substack{\uparrow \\ du\,dv}}{dA^*}$$

OK more generally
for a change of vbls
— invertible
— det > 0 (orient. pres.)
exc. poss. along fin many curves
" " pts

$$\Delta = \det\begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial x} \\ \frac{\partial U}{\partial y} & \frac{\partial V}{\partial y} \end{pmatrix} =: \frac{\partial(U,V)}{\partial(x,y)}$$

Jacobian det
stretching factor

$$\left(\frac{\partial U}{\partial x} \;\; \frac{\partial V}{\partial x} \right) = D_x(U,V)$$

$$\left(\frac{\partial U}{\partial y} \;\; \frac{\partial V}{\partial y} \right) = D_y(U,V)$$

$$\frac{\partial(U,V)}{\partial(x,y)} = \| D_x(U,V) \times D_y(U,V) \|$$

Chg of vbls:

$$\iint_{R_{x,y}} f(U(x,y),V(x,y)) \frac{\partial(U,V)}{\partial(x,y)} \, dA$$

dxdy

$$= \iint_{\substack{R^* \\ u,v}} f(u,v)\, d\underset{du\,dv}{A^*}$$

for U, V
inv. chg. of vbls,
pres. orientation.

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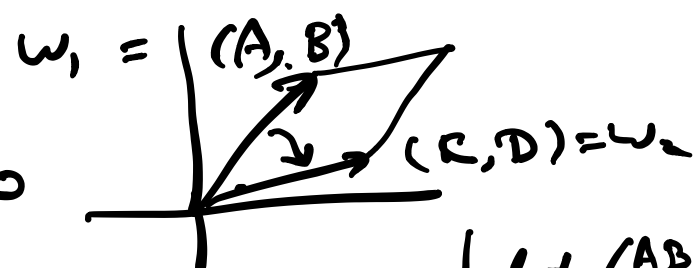
what if $\det < 0$?

Ex $(1,0) \rightsquigarrow (A,B)$
$(0,1) \rightsquigarrow (C,D)$



$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} < 0$

$\| w_1 \times w_2 \| = \text{area} = \left| \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|$

So: put in abs. val. of Jac. det.

$(x,y) \longleftrightarrow (u,v)$

$x = X(u,v), \quad y = Y(u,v)$

p394

$$\iint_{\substack{R \\ x,y}} f(x,y)\, dx\, dy = \iint_{\substack{R^* \\ u,v}} f(X(u,v), Y(u,v)) \left| \frac{\partial(X,Y)}{\partial(u,v)} \right| du\, dv$$

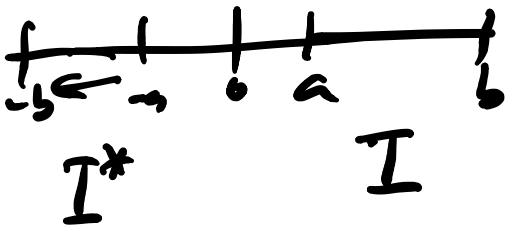
$\dfrac{\partial(X,Y)}{\partial(u,v)} = J(u,v)$

Abs val? Not in 1 dim!?

1 vbl. $\int_{x=a}^{b} \cdots dx = \int_{u=-a}^{-b} \cdots (-1)\, du$

$u = -x \qquad du = -dx$

$= -\int_{-b}^{-a} \cdots (-1)\, du$



$= \int_{-b}^{-a} \cdots du$

$\iint \cdots dA$

$dA \to dx\,dy$

$\iint \cdots dx \wedge dy$

Don't need 1-1.

Ex. $u = y, \quad v = x$

$(x,y) \mapsto (y,x)$

$\text{Jac} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 < 0.$

$$\iint_R f(x,y)\, dx \wedge dy = \iint_{R^*} f(y,x)\,(-1)\,\overbrace{dy \wedge dx}$$

$$= \iint_{R^*} f(y,x)\, dx \wedge dy$$

Ex. Cartesian $\leftrightarrow$ polar

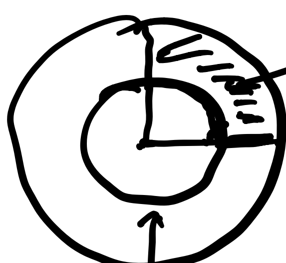
$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r > 0$$

exc at 1 pt

$$\iint_{x,y} \cdots \overbrace{dx\,dy}^{dA} = \iint_{r,\theta} \cdots \underbrace{r\,dr\,d\theta}_{dA^*}$$



$$\iint_R x\,dA$$

dxdy

polar $\quad 1 \le r \le 2$
$\quad 0 \le \theta \le \pi/2$

Get $$\int_{\theta=0}^{\pi/2}\left(\int_{r=1}^{2} (r\cos\theta)\, r\,dr\right)d\theta$$

$$\int_1^2 r^2\cos\theta \cdot dr = \frac{r^3}{3}\cos\theta \Big|_{r=1}^{2}$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right)\cos\theta = \frac{7}{3}\cos\theta$$

$$\int_{\theta=0}^{\pi/2} \frac{7}{3}\cos\theta \, d\theta$$

$$= \frac{7}{3}\sin\theta \Big|_{\theta=0}^{\pi/2} = \frac{7}{3}(1-0) = \underline{\frac{7}{3}}$$

More than 2 vbls

$$\underset{\text{open}}{S} \subset \mathbb{R}^n$$

$$f: S \to \mathbb{R}$$

$$\begin{array}{c} \cup\! \mid \\ R \end{array} \quad \text{closed region}$$

Integrate $\underline{f \text{ over}}$ R

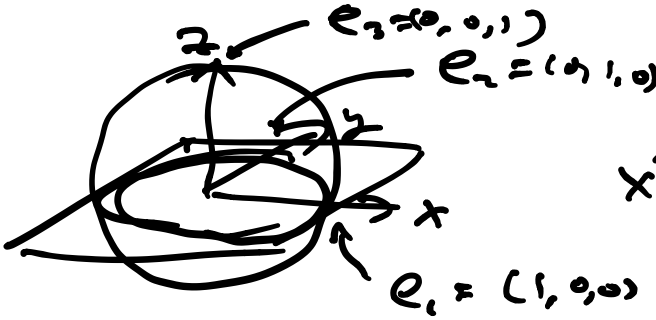
Ex $n = 3$. $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z)$

R : closed unit ball

$$x^2 + y^2 + z^2 \le 1$$

$$\iiint_R f(x, y, z)\, d\overbrace{V}^{dx\,dy\,dz}$$



$x^2+y^2+z^2 \le 1$

$$\iiint_R f\,dV = \int_{z=-1}^{1}\left(\int_{y=-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\right)dy\right)dz$$

Change of vbls:

$x, y, z \qquad u, v, w$

$x = X(u,v,w),\ y = Y(u,v,w),\ z = Z(u,v,w)$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R^*} f(X(u,v,w), Y(u,v,w), Z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Eqn 11.48 p 408

n-dims. $\int_R \cdots \int$, $\int_R$

$\underline{x} = (x_1, \ldots, x_n)$, $d\underline{x} = dx_1 \cdots dx_n$

$$\int_R f(\underline{x})\,d\underline{x}$$

Chg of vbls $\qquad \underline{x} = \underline{X}(\underline{u})$

$$\int_R f(\underline{x})\,d\underline{x} = \int_{R^*} f(\underline{X}(\underline{u}))\left|\left(\frac{\partial \underline{x}}{\partial \underline{u}}\right)\right| d\underline{u}$$

Eqn. 11.87 $\quad$ p 408

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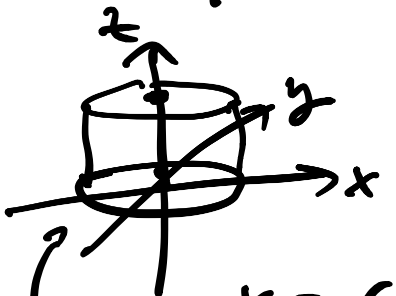
1) Cylindrical coords



x, y plane: polar r, θ

use z

r, θ, z

$x = r\cos\theta, \ y = r\sin\theta, z = z$

$x^2 + y^2 \le 1, \quad 0 \le z \le 1$

Cyl. coords:

$0 \le r \le 1, \ 0 \le z \le 1$

$0 \le \theta \le 2\pi$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r > 0$$

$$\underbrace{dx\,dy}\,dz = \underbrace{r\,dr\,d\theta}\,dz$$

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2) Spherical coords

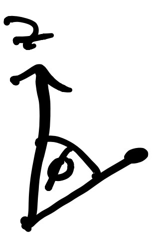
$$\rho, \underbrace{\theta, \phi}$$

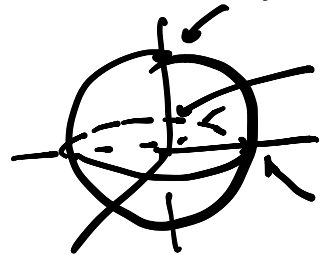
$\rho \geq 0$, dist. from O.

longitude: θ $\quad 0 \leq \theta \leq 2\pi$

latitude ϕ $\quad 0 \leq \phi \leq \pi$

$\uparrow$ N $\quad \uparrow$ S





e_3: $\rho = 1$, θ arb., $\phi = 0$

e_2: $\rho = 1$, $\theta = \pi/2$, $\phi = \pi/2$

e_1: $\rho = 1$, $\theta = 0$, $\phi = \pi/2$

unit sphere

$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2 \sin\phi$$

turp p. 411

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$

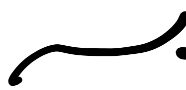
$$= \iiint_{R^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Ex's in higher dims

Ex 3 & 4, pp 411–413

Γ p184

Chap 12 Surf $\int$'s

Line $\int$'s.  $\alpha : [a,b] \to \mathbb{R}^n$

 ← curved surf

R region in plane

 flat surface

Ex. sphere $x^2+y^2+z^2=1$

hemisphere $z \geq 0$

param.

in plane

$$r : T \to S$$

$$\begin{array}{cc} \cap\!\mid & \cap\!\mid \\ \mathbb{R}^2 & \mathbb{R}^3 \\ u,v & x,y,z \end{array}$$

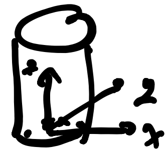
$r(u,v) = (X(u,v), Y(u,v), Z(u,v))$

X, Y, Z scalar functions

Ex. S = cylinder

$x^2+y^2=9$

$0 \le z \le 2$



Cyl. coords.

$r=3, \quad 0 \le z \le 2$

$0 \le \theta \le 2\pi$

$x = r\cos\theta$

$y = r\sin\theta$

$u = \theta, \; v = z$

$r(u,v) = (3\cos u, 3\sin u, v)$

$0 \le u \le 2\pi$

$0 \le v \le 2$

Ex, p418 par of sphere

2 p419 " " cone

Surf integral

$$\iint_S f \, dS \qquad \underset{\text{surf}}{S} \subseteq \mathbb{R}^3$$

Case: flat $S \subseteq \underset{\substack{x,y \\ z=0}}{\mathbb{R}^2} \subseteq \underset{x,y,z}{\mathbb{R}^3}$

$$r=(x,y,z): \underset{\substack{\cap \\ \mathbb{R}^2}}{T} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^3}}{S}$$

Suppress z

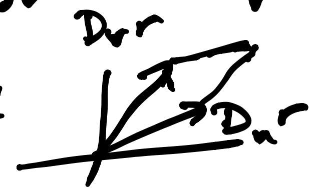
$$r=(x,y): \underset{\substack{\cap \\ \mathbb{R}^2 \\ u,v}}{T} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^2 \\ x,y}}{S}$$

Chg of vbls

$$\iint_S f(x,y)\,dA = \iint_T f(\underbrace{X(u,v)\,Y(u,v)}_{r(u,v)})\left|\frac{\partial(X,Y)}{\partial(u,v)}\right| du\,dv$$

$$\det\begin{array}{cl} \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{pmatrix} & \begin{array}{l} \leftarrow D_u r = r_u \\ \leftarrow D_v r = r_v \end{array} \end{array}$$

$\parallel$

area of 

$\parallel$

$\|r_u \times r_v\|$, mult of $\vec{k}$

For gen'l $S \subseteq \mathbb{R}^3$, do same

$$\begin{array}{ccc} r = (X,Y,Z) : T & \longrightarrow & S \\ \cap\shortmid & & \cap\shortmid \\ \mathbb{R}^2 & & \mathbb{R}^3 \\ u,v & & x,y,z \end{array}$$

param: diff, one-to-one

$$r_u = \left(\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}\right)$$

$$r_v = \left(\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}\right)$$

$r_u \times r_v \perp S$ at that pt

– normal vector to S

§12.3

Define

$$\iint_S f(x,y,z)\,dS := \iint_T f(r(u,v))\,\|r_u \times r_v\|\,du\,dv$$

Surf $\int$ (ord. dbl. $\int$)

Agrees w prev def of dbl $\int$ if $S \subseteq \mathbb{R}^2$.

$\iint_S$ indep of r. (§12.8)

Recall $S \subseteq \mathbb{R}^2$ x, y

$C = \partial S$

$\omega = P\,dx + Q\,dy$ diff 1-form

$$\oint_C \omega = \iint_S d\omega \quad \text{Green's Thm}$$

$$\rightarrow \oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

Gen'l'n: $S \subseteq \mathbb{R}^3$, $C = \partial S$

Stokes' Thm

$\omega = P\,dx + Q\,dy + R\,dz$

$$\longrightarrow \oint_C \omega = \iint_S d\omega$$

Use $dy \wedge dx = -\,dx \wedge dy$

$\longrightarrow$ $dx \wedge dx = 0$

Get

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

$\rightsquigarrow$

$$\oint_C P\,dx + Q\,dy + R\,dz$$

$$= \iint_S \Big[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz \quad \doteq \frac{\partial(y,z)}{\partial(u,v)}\, du \wedge dv$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \Big]$$

Th 12.3, p 428

Proven by using that $\iint_S$ is def via dbl $\iint_T$ in $\mathbb{R}^2$ & using Green's Thm in $\mathbb{R}^2$.

Gen'l'n to n dim's

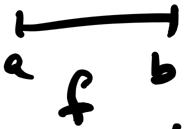
$$S \subseteq \mathbb{R}^N$$

n-dim'l $\quad \partial S$: n-1 dim'l

ω diff'l $\underline{\underline{n-1}}$ form

$$\int_{\partial S} \omega = \int_S d\omega$$

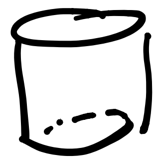
$n=1$ $\qquad$ $n-1=0$

 in $\mathbb{R}$

$$\int_a^b f'(x)\,dx = \int_a^b df = f\Big|_a^b$$ $\qquad$ 2° FTC

Key ex of surf $\int$:

$$\iint_S 1\, dS = \text{surf area of } S$$

Ex. Cylinder S  $x^2+y^2=9$, $0 \le z \le 2$

$r(u,v) = (3\cos u, 3\sin u, v)$ ←

$0 \le u \le 2\pi$, $0 \le v \le 2$

$r_u = (-3\sin u, 3\cos u, 0)$

$r_v = (0, 0, 1)$

$\|r_u \times r_v\| = \|(3\cos u, 3\sin u, 0)\| = 3$

surf area =

$$\int_{v=0}^{2} \underbrace{\int_{u=0}^{2\pi} 3\, du}_{6\pi}\, dv = 12\pi$$

Ex 1 pp 427–428

hemisphere

Ex S is graph of

$$z = g(x,y)$$

over $T \subseteq \mathbb{R}^2_{x,y}$

Param $x = u,\ y = v,\ z = g(u,v)$

T in u,v plane

$$r(u,v) = (u, v, g(u,v))$$

$$\|r_u \times r_v\| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

$$\iint_S f\, dS = \iint_T f\left(\sqrt{}\right) dA$$

In partic: $f = 1$

Surf area of graph of $z = g(x,y)$ over $T \subseteq \mathbb{R}^2$ is

$$\iint_S 1\, dS = \iint_T \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Analogous to: arc length of graph of $y = g(x)$ over $[a,b]$:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$S \subset \mathbb{R}^3$ $\quad z$ implicit fn of x, y

$F(x, y, z) = 0$

over $T \subseteq \mathbb{R}^2$

area (S)

$$= \iint_T \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}{\left|\frac{\partial F}{\partial z}\right|}\, dx\, dy$$

(12.12), p 427

Generalizes prev.

$$F(x, y, z) = z - g(x, y)$$

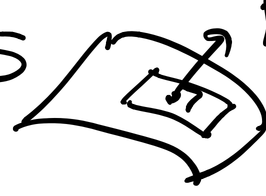
Vector fields

$$\begin{tikzcd} r: T \arrow[r] \arrow[d, phantom, "\cap"] & S \arrow[d, phantom, "\cap"] \\ \mathbb{R}^2 & \mathbb{R}^3 \end{tikzcd}$$

surface

(u, v) $\quad x, y, z$ $\quad$ diff, one-to-one, invertible

S



$N \perp S$

$N := r_u \times r_v \neq 0$

$\|N\| = \|r_u \times r_v\|$

$=$ dilation factor

$\underline{n} = \dfrac{N}{\|N\|}$ has norm 1 (unit vector)

$\underline{n}$: unit normal vector

F v. fld. on $\mathbb{R}^3$

On S, $F\cdot\underline{n}$: scalar

$$\iint_S F\cdot\underline{n}\,dS =: \iint_S F\cdot d\underline{S}$$

$F\cdot\underline{n}$: component of F in direction of $\underline{n}$.

flux

$$\iint_S F\cdot d\underline{S} = \iint_S F\cdot\underline{n}\,dS$$

$$= \iint_S F\cdot\frac{N}{\|N\|}\,dS$$

$$= \iint_T F\cdot\frac{N}{\|N\|}\,\underbrace{\|r_u\times r_v\|}_{\to\|N\|}\,du\wedge dv$$

$$= \iint_T F\cdot(r_u\times r_v)\,dA$$

"flux int. of F over S"

$\to$ $F = \underline{P}\,\vec{i} + \underline{Q}\,\vec{j} + \underline{R}\,\vec{k}$

$P, Q\ R : S \longrightarrow \mathbb{R}$

$$\vec{r}_u \times \vec{r}_v = \frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}$$

$$r(u,v) = X(u,v)\vec{i} + Y(u,v)\vec{j} + Z(u,v)\vec{k}$$

$$\iint_S F \cdot d\underline{S} = \iint_S F \cdot \underline{n}\, dS = \iint_T F \cdot (r_u \times r_v)\, du \wedge dv$$

$$= \iint_T \underbrace{P}_{P(r(u,v))} \frac{\partial(y,z)}{\partial(u,v)} + \iint_T Q \frac{\partial(z,x)}{\partial(u,v)} + \iint_T R \frac{\partial(x,y)}{\partial(u,v)}$$

$du \wedge dv$

$$= \iint_S P\, dy \wedge dz + \iint_S Q\, dz \wedge dx + \iint_S R\, dx \wedge dy$$

(1) $F \cdot \underline{n}\, dS = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$

$* \cdot n\, dS$

$\vec{i} \rightsquigarrow dy \wedge dz$ diff'l

$\vec{j} \rightsquigarrow dz \wedge dx$ 2-form

$\vec{k} \rightsquigarrow dx \wedge dy$

$\omega = P\, dx + Q\, dy + R\, dz$ diff 1-form

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz \leftarrow$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx \leftarrow \quad (2)$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \leftarrow$$

Corresp v. field: $F = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\to \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \text{curl } F$$

$$F \longleftrightarrow \omega$$

$$\text{curl } F \cdot \underline{n} \, dS = d\omega$$

$$\text{curl } F = \nabla \times F$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \nabla f$$

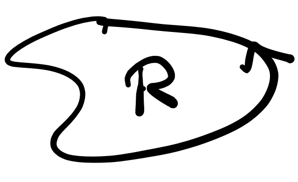
$$\nabla \times F = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

Ex. In $\mathbb{R}^2$, x, y plane.

$$F = P\vec{i} + Q\vec{j}$$

fns of x, y

$$\to \text{curl } F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$



C $\quad \alpha : [a, b] \to C$

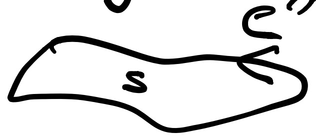
param.

Green's thm

$$\oint_C F \cdot d\alpha = \iint_R \overbrace{(\text{curl } F) \cdot \vec{k}}^{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} \, dA$$

More gen:

$\partial S = C$



$\subset \mathbb{R}^3$

(p 441)

Stokes thm

$$\oint_C F\cdot d\alpha = \iint_S (\text{curl } F)\cdot \underline{n}\, dS$$

$$= \iint_S \text{curl} \cdot d\underline{S}$$

$$\text{curl } (\nabla f) = 0$$

fn on $\mathbb{R}^3$

$$\underbrace{\nabla \times \nabla}_{0} f = 0$$

$$\to d^2 f = d(df) = 0 \leftarrow$$

---

curl F — physically

F v. field on subset of $\mathbb{R}^3$

"flow"

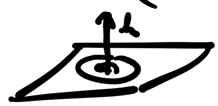
curl F : axis of rotation

Ex 1.

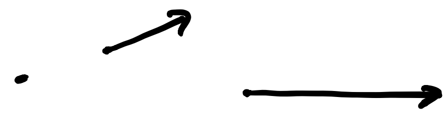
$F = -y\vec{i} + x\vec{j}$

$P = -y, \quad Q = x, \quad R = 0.$

$$\text{curl } F = (1-(-1))\vec{k} = 2\vec{k}$$



Ex 2. $F = x\vec{i} + y\vec{j}$



$P = x \quad Q = y$

$$\text{curl } F = (0-0)\vec{k} = 0$$

no rotation

$F = P\vec{i} + Q\vec{j}$ on S

Recall

F conservative

$\Updownarrow$

$F = \nabla f$

$\Downarrow \quad \Uparrow$ S simply conn

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Now $\rightarrow$ $\Updownarrow$

$$\text{curl } F = 0$$

This generalizes to F on $\mathbb{R}^3$

V field $F = P\vec{i} + Q\vec{j} + R\vec{k}$

on $\mathbb{R}^3$ (or region)

P, Q, R cont. diff.

F is conservative

$\Updownarrow$

$F = \nabla f$ (some f)

$\Downarrow$ $\quad$ $\Uparrow$ ← if region is simply conn.

$\operatorname{curl} F = 0$

$\operatorname{curl}(\nabla f) = 0$

$\parallel$

$\nabla \times \nabla f$

reason: a loop C bounds surf S

$$\partial S = C$$

Stokes:

$$0 \stackrel{?}{=} \oint_C F \cdot d\alpha = \iint_S \underbrace{\operatorname{curl} F}_{= 0} \cdot d\underline{S}$$

---

divergence

$$F = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

scalar field : $\mathbb{R}^3 \to \mathbb{R}$

(also $\mathbb{R}^n$)

$$\operatorname{div} F = \nabla \cdot F$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

---

div (curl F) = 0

$$\nabla \cdot (\nabla \times F) = 0$$

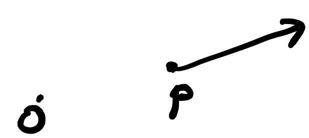
Physical interp:

F flow

div F: extent to which it flows away

Ex. 1) In plane:

$$F = \underset{P}{\underset{\uparrow}{x}}\vec{i} + \underset{Q}{\underset{\uparrow}{y}}\vec{j}$$



$$\operatorname{div} F = 1 + 1 = 2 > 0$$

2) In plane

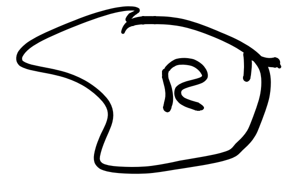
$$F = -\underset{\uparrow}{y}\,\vec{i} + \underset{\uparrow}{x}\,\vec{j}$$

$$\operatorname{div} F = 0 + 0 = 0$$

Applic of Green's Thm

2 dim'l divergence thm.

region



$C = \partial R$

F v. field on R

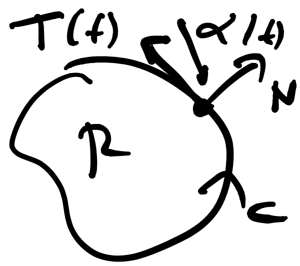
$F = P\vec{i} + Q\vec{j}$

C, param

$$\alpha : [a, b] \longrightarrow C \subseteq \mathbb{R}^2$$

$$\alpha(t) = (X(t), Y(t)) = X(t)\vec{i} + Y(t)\vec{j}$$

Tan vector to C at $\alpha(t)$



$$T(t) = \frac{d}{dt}\alpha(t) = X'(t)\vec{i} + Y'(t)\vec{j}$$

Normal vector $N(t) = Y'(t)\vec{i} - X'(t)\vec{j}$

$$\|N(t)\| = \sqrt{Y'(t)^2 + X'(t)^2} = \|\alpha'(t)\| \leftarrow$$

$\underline{n}(t) = N(t)/\|N(t)\|$, unit normal vector.

$\underline{\text{Thm}}$ $$\oint_C F \cdot \underline{n}\, ds = \underbrace{\iint_R \operatorname{div} F \, dA}$$

∂R — C

total flux of F across S

Pf. (using Green's Thm)

$$\oint_C F \cdot \underline{n}\, dS = \int_{t=a}^{b} (\underline{F} \cdot \underline{n})(t) \, \|\alpha'(t)\| \, dt$$

$$= \int_a^b \left(\frac{P\, y'(t)}{\|\alpha'(t)\|} - \frac{Q\, x'(t)}{\|\alpha'(t)\|} \right) \|\alpha'(t)\| \, dt$$

$$= \int_a^b \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt$$

$$= \oint_C P\, dy - Q\, dx$$

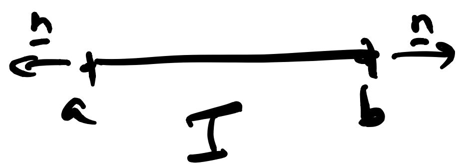
$$= \oint_C -Q\, dx + P\, dy$$

$$= \iint_R \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}_{\operatorname{div} F} dA$$

Also in other dims

Dim 1 $\qquad F$ on $\mathbb{R}^1$ $\qquad \vec{i}$

$$F = f(x)\,\vec{i}$$
$$\operatorname{div} F = f'(x)$$



$$\int_I \operatorname{div} F \, dx = \int_a^b f'(x)\, dx = f\Big|_a^b = f(b) - f(a)$$
$$= f\cdot n \Big|_{\partial I}$$

---

3 dim div. thm (Gauss's Thm)

Thm 12.6 p 457

F v. field on $\mathbb{R}^3$ cont diff

V closed region

$\partial V = S$ $\qquad \underline{n}$ unit normal

$$\uparrow$$
$$\frac{\underline{N}}{\|N\|} \leftarrow r_u \times r_v$$

Thm $$\iint_{\partial V} F\cdot \underline{n}\, dS = \iiint_V \operatorname{div} F \, dV$$

$\partial V \to S$

$$\iint_S F \cdot d\underline{S}$$

Pf similar to pf of Green's Thm

— Sum of three terms.

Show corresp terms are =.

— iterated $\int$'s

See §12.19

Holds in n dims

Ex S <u>unit</u> sphere

∂B " ball

$B = V$

$x^2+y^2+z^2=1.$

$\rightarrow F = x\vec{i} + y\vec{j} + z\vec{k}$

$\rightarrow \underline{n} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$

$$\iint_S F \cdot \underline{n}\, dS = \iint_S \underbrace{(x^2+y^2+z^2)}_{1}\, dS$$

$$= \iint_S 1\, dS = \text{area}(S)$$

$$= 4\pi$$

Sph, rad = r

$4\pi r^2$

$$\operatorname{div} F = 1 + 1 + 1 = 3$$

$$\iiint_B \operatorname{div} F \, dV = \iiint_B 3 \, dV$$

$$= 3 \operatorname{vol}(B)$$

$$= 3\left(\tfrac{4}{3}\pi\right)$$

$$= 4\pi$$

Ball, rad $= r$

vol: $\frac{4}{3}\pi r^3$