Math 370  
Problem Set #10  
Due week of Nov. 12, 2007, in lab.

Reminder: Exam #2 will take place in class on Monday, November 12, on the material covered in class since the previous exam, and up through the end of Chapter 4 of the text. Read Hoffman and Kunze, Chapter 4, Section 5 and Chapter 5, Section 1.

1. From Hoffman and Kunze, Chapter 4, do these problems: Page 134, #2(b). Page 139, #1, 5-8. [In problems 5-8, use the definition given just before problem 5. In problem 7, use problem 6.]

2. Let \(a(x), b(x), c(x) \in F[x]\), where \(F\) is a field. Suppose that \(a(x)\) and \(b(x)\) are relatively prime, and also that \(a(x)\) divides \(b(x)c(x)\). Show that \(a(x)\) divides \(c(x)\). [Hint: Do this analogously to the proof of Theorem 8 of Hoffman and Kunze, §4.5.]

3. Let \(a(x), b(x) \in F[x]\), where \(F\) is a field. Let \(d\) be the largest degree of any polynomial that divides both \(a(x)\) and \(b(x)\).
   a) Show that there is a unique monic polynomial \(g(x)\) of degree \(d\) such that \(g(x)\) divides both \(a(x)\) and \(b(x)\).
   b) Show that this polynomial \(g(x)\) is the greatest common divisor of \(a(x)\) and \(b(x)\); i.e. \(g(x)\) is the monic generator of the ideal \((a(x), b(x))\).

4. a) Let \(f(x), g(x) \in F[x]\), where \(F\) is a field. Suppose that the only polynomials in \(F[x]\) that divide both \(f(x)\) and \(g(x)\) are constant polynomials. Show that \(\{af + bg \mid a, b \in F[x]\}\) is all of \(F[x]\).
   b) Show that if \(F[x]\) is replaced by \(F[x, y]\), then the conclusion in (a) is no longer true.

5. a) Find a surjective homomorphism of \(\mathbb{R}\)-algebras \(\mathbb{R}[x] \to \mathcal{B}\) whose kernel is the ideal generated by \(x + 1\).
   b) Do the same with \(x + 1\) replaced by \(x^2 + 1\).