Read Herstein, Chapter 3, sections 3-4.

1. From Herstein, Chapter 3, do these problems:
   a) Section 3.2, page 130: #6, 7, 13 (note that #13 asks for a new proof of p.24, #15).
   b) Section 3.4, page 135: #3, 5-8.

2. Which of the following are ring homomorphisms? For those that are not, why not? For those that are, find the kernel and image.
   i) \( \mathbb{R}[x] \to \mathbb{C}, f(x) \mapsto f(3) \).
   ii) \( \mathbb{R}[x] \to \mathbb{C}, f(x) \mapsto f(2i) \).
   iii) \( \mathbb{C} \to \mathbb{R}, a + bi \mapsto a \) for \( a, b \in \mathbb{R} \).
   iv) \( \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}], a + b\sqrt{2} \mapsto a + b\sqrt{3} \) for \( a, b \in \mathbb{Q} \).
   v) \( \mathbb{Q}[\zeta] \to \mathbb{Q}[\zeta], a + b\zeta \mapsto a + b\zeta^2 \), for \( a, b \in \mathbb{Q} \), where \( \zeta = e^{2\pi i/3} \).
   vi) \( \mathbb{Z}[i] \to \mathbb{Z}/5, a + bi \mapsto a + 2b \), for \( a, b \in \mathbb{Z} \).
   vii) \( \mathbb{C} \to M_2(\mathbb{R}), a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \), for \( a, b \in \mathbb{R} \).

3. Suppose that \( \phi : \mathbb{R} \to \mathbb{R} \) is a homomorphism of rings.
   a) Show that \( \phi(r) = r \) for all \( r \in \mathbb{Z} \).
   b) Do the same for all \( r \in \mathbb{Q} \).
   c) Show that if \( r \geq 0 \) then \( \phi(r) \geq 0 \). [Hint: \( r \geq 0 \iff r = s^2 \) for some \( s \).]
   d) Show that \( \phi \) is an increasing function. [Hint: Part (c).]
   e) Conclude that \( \phi \) is the identity. [Hint: Parts (b) and (d).]

4. Let \( \mathbb{H} \) be the ring of quaternions \( \alpha = a + bi + cj + dk \), with \( a, b, c, d \in \mathbb{R} \). Define the conjugate \( \bar{\alpha} = a - bi - cj - dk \), and the absolute value \( |\alpha| \geq 0 \) by \( |\alpha|^2 = a^2 + b^2 + c^2 + d^2 \).
   a) Show that \( |\alpha|^2 = \alpha \bar{\alpha} \) and that \( \overline{\alpha \beta} = \beta \bar{\alpha} \). Conclude that \( |\alpha \beta| = |\alpha||\beta| \). Also, find all \( \alpha \in \mathbb{H} \) such that \( |\alpha| = 0 \).
   b) Show that \( \mathbb{H} \) does not have any zero-divisors. [Hint: Use part (a).]