Read Artin, Chapter 2, sections 1-5.

1. From Artin, Chapter 2, do these problems (pages 69-77):
   2.4, 2.6, 4.2, 4.3, 4.6.

2. For each permutation $p$ of $\{1, \ldots, n\}$, let $M_p$ denote the associated permutation matrix. Show that if $p_1 \circ p_2 = p_3$ then $M_{p_1} M_{p_2} = M_{p_3}$.

3. The initial configuration in the 15 puzzle consists of the numbers 1, ..., 15 in that order (reading across successive lines of the $4 \times 4$ matrix), and a blank in the lower right hand corner. Consider an arbitrary configuration in the 15 puzzle in which the blank square is in the lower right hand corner. Show that if this configuration can be taken in $n$ moves to the initial configuration, then $n$ is even. Deduce that in this case the given configuration represents an even permutation of $\{1, \ldots, 15\}$. [Hint: Regard each square as labeled by a + or − in a checkerboard pattern, and regard each move as a transposition in the symmetric group $S_{16}$. Also regard $S_{15}$ as the subgroup of $S_{16}$ in which 16 is fixed.]

4. Determine which of the following are groups under matrix multiplication:
   a) The set of $n \times n$ real matrices with positive determinant.
   b) The set of $n \times n$ complex matrices whose determinant has absolute value $k$. (The answer to this part depends on the value of the real number $k$.)
   c) The set of $2 \times 2$ matrices of the form $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, where $\theta \in \mathbb{R}$.
   d) The set of $2 \times 2$ complex matrices that either have the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ or have the form $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, where $a \in \mathbb{C}$ is non-zero.

5. Define the order of an element $g$ in a group $G$ to be the smallest positive integer $r$ such that $g^r = 1$; or $\infty$ if there is no such $r$.
   a) For those items in problem 4 that are groups, find all the elements of finite order. (For parts a,b of that problem, just take $n = 1$.)
   b) In the group $GL_n(\mathbb{R})$, let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, and $C = AB$. Find the orders of the three elements $A, B, C$.

6. Let $a, b$ be elements of a group $G$.
   a) If $a, b \in G$ each have finite order, must $ab$ have finite order? Give either a proof or a counterexample.
   b) Does your answer to part (a) change if $G$ is assumed to be abelian?