1. Let $V$ be the real vector space of polynomials $f(x)$ of degree at most 3. Define $T : V \to V$ by $T(f) = \frac{1}{x} \int_0^x f(t) \ dt$.

   a) Show that $T$ is a linear transformation.
   
   b) Find the matrix of $T$ with respect to the basis $\{1, x, x^2, x^3\}$ of $V$.
   
   c) Find the kernel and image of $T$, and the rank and the nullity of the associated matrix.

2. Let $T : V \to V$ be a linear operator on a vector space $V$.

   a) Show that the kernel of $T$ is contained in the kernel of $T^2 = T \circ T$.
   
   b) Can the two kernels be equal? Can they be unequal?

3. Let $F$ be a field. For each of the following, either give an example or explain why no such example exists:

   a) A linear transformation $F^2 \to F^2$ whose kernel is one-dimensional.
   
   b) A linear transformation $F^4 \to F^3$ whose kernel is trivial.
   
   c) A linear transformation $F^3 \to F^2$ taking $(1, 2, 3)$ to $(4, 5)$.

4. Suppose that $A$ and $B$ are similar $n \times n$ matrices over a field $F$ (i.e. $B = C^{-1}AC$ for some $C \in \text{GL}_n(F)$).

   a) Show that $A^n$ and $B^n$ are similar.
   
   b) Show that if $f(x) \in F[x]$, then $f(A)$ and $f(B)$ are similar.
   
   c) Show that if $f(x) \in F[x]$ and $f(A) = 0$, then $f(B) = 0$.

5. a) Let $V, W, X$ be finite dimensional vector spaces, and let $S : W \to X$ and $T : V \to W$ be linear transformations. Show that $\text{rank}(S \circ T) \leq \text{rank}(T)$ and that $\text{rank}(S \circ T) \leq \text{rank}(S)$.

   b) Let $T : V \to V$ be a linear operator on a finite dimensional vector space $V$. Let $r_n$ be the rank of $T^n = T \circ T \circ \cdots \circ T$ (with $n$ copies of $T$). Prove that $r_{n+1} \leq r_n$ for all $n$, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant.

   c) What happens in (b) if instead $V$ is infinite dimensional?

6. a) Find polynomials $f_0, f_1, f_2$ of degree $\leq 2$ in $\mathbb{R}[x]$ such that for all $0 \leq i, j \leq 2$ we have that $f_i(j) = 1$ if $i = j$ and $f_i(j) = 0$ if $i \neq j$.

   b) Prove that $f_0, f_1, f_2$ form a basis of $P_2 = \{\text{polynomials of degree } \leq 2\}$. 