This exam consists of ten problems. Do them all, showing your work and explaining your assertions. Give yourself 80 minutes.

1. Consider the system of equations

\[
\begin{align*}
x - 2y - z &= 5 \\
x + 3y + 2z &= 1 \\
2x + y + z &= c,
\end{align*}
\]

where \( c \in \mathbb{R} \). For each value of \( c \), determine how many solutions \((x, y, z)\) the system of equations has.

2. Let \( G \) be a group and \( g \in G \). Show that the set \( \{1, g, g^2, g^3, \ldots\} \) is a subgroup of \( G \) if and only if \( g \) has finite order.

3. Let \( H \) be the set of \( n \times n \) complex matrices \( M \) such that \(|\det M| = 1\). Show that \( H \) is a subgroup of \( \text{GL}_n(\mathbb{C}) \). Is \( H \) normal in \( \text{GL}_n(\mathbb{C}) \)?

4. Show that the set of complex numbers of the form \( a + bi \), with \( a, b \in \mathbb{Q} \), is a field.


6. a) Show that the set of continuous functions \( f : \mathbb{R} \to \mathbb{R} \) is a vector space over the field of scalars \( \mathbb{R} \).

   b) Determine whether this vector space is finite or infinite dimensional.

7. Find all integers \( n \) such that the dihedral group \( D_{11} \) has an element of order \( n \).

8. Let \( V \subset \mathbb{R}^3 \) be the set of vectors \((x, y, z)\) with \( x + y + z = 0 \).

   a) Show that \( V \) is a subspace of \( \mathbb{R}^3 \).

   b) Find a basis of \( V \) and find the dimension of \( V \).

9. Suppose that \( f : G \to H \) and \( g : H \to K \) are injective group homomorphisms. Prove that \( g \circ f : G \to K \) is also an injective group homomorphism.

10. a) Show that in \( \mathbb{R}^3 \), the vectors \((0, 1, 1), (1, 1, 0), (1, 1, 1)\) form a basis.

    b) Express the vector \((1, 2, 3)\) in terms of this basis.