Math 502 Problem Set #2 Due week of Sept. 11, 2017, in lab.

Read Artin, Chapter 2, sections 1,2,5,6.

1. From Artin, Chapter 2, do these problems (pages 69-77):
2.4, 2.6, 5.4, 6.2.

2. In the notation of problem 2.6 on page 70 of Artin (assigned above), determine which of the following maps are homomorphisms, and which are isomorphisms. (Your answer may depend on the properties of the group.)

a) $\phi : G \to G^\circ$ is defined by $\phi(a) = a$.

b) $\psi : G \to G^\circ$ is defined by $\psi(a) = a^{-1}$.

3. Determine which of the following are groups under matrix multiplication:

a) The set of $n \times n$ real matrices with positive determinant.

b) The set of $n \times n$ complex matrices whose determinant has absolute value k. (The answer to this part depends on the value of the real number k.)

c) The set of 2×2 matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$.

d) The set of 2×2 complex matrices that either have the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ or have the form $\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}$, where $a \in \mathbb{C}$ is non-zero.

4. Define the *order* of an element g in a group G to be the smallest positive integer r such that $g^r = 1$; or ∞ if there is no such r.

a) For those items in problem 3 that are groups, find all the elements of finite order. (For parts a,b of that problem, just take $n = 1$.)

b) In the group $\mathrm{GL}_n(\mathbb{R})$, let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, and $C = AB$. Find the orders of the three elements A, B, C.

5. Let a, b be elements of a group G.

a) If $a, b \in G$ each have finite order, must ab have finite order? Give either a proof or a counterexample.

b) Does your answer to part (a) change if G is assumed to be abelian?