Math 502 Problem Set #5 Due week of Oct. 2, 2017, in lab.

(Note: Due to fall break, those in the Thursday lab can attend the Tuesday lab. Their problem set can be submitted to the TA by 10:30am on Wednesday or submitted in class on Wednesday.)

Read Artin, Chapter 2, sections 9-12.

1. From Artin, Chapter 2, do these problems (pages 69-77):
8.5, 8.8, 9.1-9.3, 11.3.

2. Define the *center* of a group G to be $Z = \{g \in G \mid (\forall h \in G)\, gh = hg\}$.
   a) Is Z a subgroup? Is it normal?
   b) Find the center of $C_n, D_n, S_n, \mathbb{Z}, SL_2(\mathbb{R})$.

3. Let G be a group and let H be a subgroup of G. For every left coset L of H in G, let $\phi(L) = \{g \in G \mid g^{-1} \in L\}$.
   a) Show that $\phi(L)$ is a right coset of H in G.
   b) Show that ϕ defines a bijection between the set of left cosets of H in G and the set of right cosets of H in G.

4. Let m and n be relatively prime integers.
   a) Using the isomorphism $\mathbb{Z}/mn\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (for such m and n), prove that for any integers a and b there is a unique integer c such that $c \equiv a \pmod{m}$, $c \equiv b \pmod{n}$, and $0 \leq c < mn$. [This is called the Chinese Remainder Theorem.]
   b) In the case that $m = 5$, $n = 7$, $a = 3$, and $b = 2$, find c.

5. Let G be a group.
   a) Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, and hence there is a quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. This quotient group is called the *outer automorphism group* of G, and we denote it by $\mathrm{Out}(G)$. Show that $\mathrm{Out}(G)$ is the trivial group (of one element) if and only if every automorphism of G is inner.
   b) For $G = C_3$, find the groups $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$, and $\mathrm{Out}(G)$ (in particular finding their orders). Do the same for $G = S_3$.