Math 502 Problem Set #12 Due week of Nov. 20, 2017, in lab.

Read Artin, Chapter 7, sections 1-7.

1. From Artin, Chapter 7, do problems 1.1, 1.2, 2.2, 2.3, 3.2, 5.2 (pages 221-228).

2. Define the *complex upper half plane* $\mathbb{H}$ to be the set of all complex numbers whose imaginary part is positive.

a) Show that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by $A \cdot z = (az+b)/(cz+d)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

b) Let $N = \{\pm I\}$. Show that N is a normal subgroup of $\mathrm{SL}_2(\mathbb{R})$, and that an element $A \in \mathrm{SL}_2(\mathbb{R})$ leaves *every* element of $\mathbb{H}$ fixed if and only if $A \in N$.

c) Define $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/N$. Show that there is a unique action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ such that $\alpha \cdot z = A \cdot z$ for $z \in \mathbb{H}$, if $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ is the image of $A \in \mathrm{SL}_2(\mathbb{R})$ (where the right hand side is given by the action of $\mathrm{SL}_2(\mathbb{R})$).

d) Is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ faithful? transitive? What about the action of $\mathrm{PSL}_2(\mathbb{R})$?

[*Remark*: If $\mathbb{H}$ is given the geometric structure for which $\mathrm{PSL}_2(\mathbb{R})$ is the group of symmetries, then the upper half plane $\mathbb{H}$ becomes a non-Euclidean space.]

3. a) Find the smallest integer n such that D_7 is isomorphic to a subgroup of S_n. Prove your assertion.

b) Find the smallest integer n such that C_6 is isomorphic to a subgroup of S_n. Prove your assertion. [Hint: n may be smaller than you think.]

4. a) Find all the subgroups of A_4. For each, find its order, find its normalizer, and find the number of conjugate subgroups. Also, for each, verify the counting formula applied to the action of A_4 on its subgroups.

b) What does the class equation say, numerically, in the case of the group A_4?

5. Let G be a simple group; i.e., the only normal subgroups of G are itself and the trivial group.

a) Show that if G is abelian then G is cyclic of prime order.

b) Show that if G is non-abelian, then the order of G is divisible by at least two distinct prime numbers.