Math 502 Problem Set #15 Due in class on Monday, Dec. 11, 2017

Reminder: There will be an exam in class on Monday, Dec. 11. This problem set is a sample exam.

1. Let G be a group of order 27. Prove that every proper subgroup of G is abelian.

2. Let V be a finite dimensional vector space over a field F. Let $B : V \times V \to F$ be a symmetric bilinear pairing, and define $\phi : V \to V^*$ by $\phi(v) : w \mapsto B(v, w)$. Show that ϕ is a homomorphism; and that ϕ is an isomorphism if and only if B is non-degenerate.

3. a) Explicitly describe a non-trivial group action of C_2 on C_{12}.
   b) For which elements of C_{12} is the stabilizer non-trivial under that group action?

4. Give an example of a matrix $A \in M_2(\mathbb{R})$ such that there is *no* orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A, but such that there *is* an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A (viewed as a matrix in $M_2(\mathbb{C})$).

5. Let $\sigma \in S_5$ be the element whose disjoint cycle structure is $(1, 2, 3)(4, 5)$.
   a) Determine how many elements of S_5 are conjugate to σ.
   b) Find the order of the centralizer of σ in S_5.

6. For any subgroup G of $\mathrm{GL}_2(\mathbb{R})$, let G act on $\mathbb{R}^2$ by matrix multiplication (i.e., $A \cdot v = Av$ for $A \in G$ and $v \in \mathbb{R}^2$, viewing v as a column vector).
   a) Find the orbit of the first standard basis vector $e_1 \in \mathbb{R}^2$ under the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$.
   b) Do the same for the action of the orthogonal group O_2 on $\mathbb{R}^2$.

7. Find all groups of order 99, up to isomorphism.

8. Find the number of irreducible representations of D_{11} (up to isomorphism) and find their dimensions.