Math 503 Problem Set #2 Due the week of Jan. 22, 2007, in lab.

Read Artin, Chapter 10, sections 4-7.

Part A:
From Artin, do these problems (given at the end of Chapter 10):
Section 10.3: 23, 30 (in #30, IJ denotes the product ideal of I and J); Section 10.4: 2; Section 10.6: 1; Section 10.7: 1, 2(a,b).

Part B:
1. Which of the following commutative rings are integral domains? Which are fields? $\mathbb{Z}/15\mathbb{Z}$, $\mathbb{R}[x]/(x^2+2)$, $\mathbb{R}[x]/(x^2-2)$, $\mathbb{F}_2[x]/(x^2-1)$, $\mathbb{F}_2[x]/(x^2-x-1)$, $\mathbb{Z}[x]/(2,x)$, $\mathbb{Z}[x]/(2-x)$, $\mathbb{R}[x,y]/(y-x^2)$, $\mathbb{R}[x,y]/(x,y-x^2)$, $\mathbb{R}[x,y]/(y,y-x^2)$.

2. In $\mathbb{Z}$, let $I=(m)$ and $J=(n)$, where m,n are positive integers.
a) Find the ideals IJ, $I+J$, $I\cap J$ in each of these two cases:
i) $m=3$, $n=5$.
ii) $m=6$, $n=10$.
b) More generally, describe the ideals IJ, $I+J$, $I\cap J$ in terms of the integers m,n. Prove your assertions.

3. Prove, or disprove and salvage: If K is a field, and $f(x)\in K[x]$ is a polynomial of degree at least 1, then $K[x]/(f(x))$ is a field if and only if $f(x)$ has no roots in K.

Part C:
From Artin, do these problems (at the end of Chapter 10):
Section 10.3: 26, 32; Section 10.4: 7.

Also do the following problem:
a) If a,b,c are non-zero elements of a ring R, we say that $a=bc$ is a *non-trivial factorization* of a if neither b nor c is a unit in R. Which elements of $\mathbb{Z}$ can be factored non-trivially? Which elements of $\mathbb{R}[x]$?
b) Find all the units in the ring $\mathbb{Z}[i]$ of Gaussian integers.
c) Which of the following elements of $\mathbb{Z}[i]$ can be factored non-trivially? For each one that can be, do so explicitly. $2,3,5,7,11,13,15,3i,5i,2+i,3+i$
d) Make a conjecture about which Gaussian integers can be factored non-trivially.