Math 503

Problem Set #2

Due the week of Jan. 22, 2007, in lab.

Read Artin, Chapter 10, sections 4-7.

Part A:
From Artin, do these problems (given at the end of Chapter 10):
Section 10.3: 23, 30 (in #30, \(IJ\) denotes the product ideal of \(I\) and \(J\)); Section 10.4: 2; Section 10.6: 1; Section 10.7: 1, 2(a,b).

Part B:
1. Which of the following commutative rings are integral domains? Which are fields?
\[\mathbb{Z}/15\mathbb{Z}, \mathbb{R}[x]/(x^2 + 2), \mathbb{R}[x]/(x^2 - 2), \mathbb{F}_2[x]/(x^2 - 1), \mathbb{F}_2[x]/(x^2 - x - 1), \mathbb{Z}[x]/(2, x), \mathbb{Z}[x]/(2 - x), \mathbb{R}[x, y]/(y - x^2), \mathbb{R}[x, y]/(x, y - x^2), \mathbb{R}[x, y]/(y, y - x^2).\]

2. In \(\mathbb{Z}\), let \(I = (m)\) and \(J = (n)\), where \(m, n\) are positive integers.
   a) Find the ideals \(IJ, I + J, I \cap J\) in each of these two cases:
      i) \(m = 3, n = 5.\)
      ii) \(m = 6, n = 10.\)
   b) More generally, describe the ideals \(IJ, I + J, I \cap J\) in terms of the integers \(m, n\). Prove your assertions.

3. Prove, or disprove and salvage: If \(K\) is a field, and \(f(x) \in K[x]\) is a polynomial of degree at least 1, then \(K[x]/(f(x))\) is a field if and only if \(f(x)\) has no roots in \(K\).

Part C:
From Artin, do these problems (at the end of Chapter 10):
Section 10.3: 26, 32; Section 10.4: 7.

Also do the following problem:
a) If \(a, b, c\) are non-zero elements of a ring \(R\), we say that \(a = bc\) is a non-trivial factorization of \(a\) if neither \(b\) nor \(c\) is a unit in \(R\). Which elements of \(\mathbb{Z}\) can be factored non-trivially? Which elements of \(\mathbb{R}[x]\)?
b) Find all the units in the ring \(\mathbb{Z}[i]\) of Gaussian integers.
c) Which of the following elements of \(\mathbb{Z}[i]\) can be factored non-trivially? For each one that can be, do so explicitly. 2, 3, 5, 7, 11, 13, 15, 3i, 5i, 2 + i, 3 + i
d) Make a conjecture about which Gaussian integers can be factored non-trivially.