Math 503 Problem Set #3 Due the week of Jan. 29, 2007, in lab.

Read Artin, Chapter 10, sections 5-8.

Part A:
From Artin, do these problems (given at the end of Chapter 10):
Section 10.5: 2, 4, 6; 10.7: 3, 7, 10 (and relate your answer to the last problem on PS #1).

Part B:
1. Let \( R \) be a commutative ring. Show that if \( R \) is an integral domain then the characteristic of \( R \) is either 0 or prime. Does the converse hold?

2. Let \( R \) be a commutative ring and \( f(x) \in R[x] \) a polynomial of degree \( n > 0 \).
   a) Show that if \( R \) is an integral domain then \( f(x) \) has at most \( n \) roots in \( R \). [Hint: Use induction to show that if \( a_1, \ldots, a_r \in R \) are distinct roots of \( f(x) \), then the product \( (x - a_1) \cdots (x - a_r) \) divides \( f(x) \).]
   b) Show by example that the same assertion need not hold if \( R \) is not an integral domain. [Hint: Try \( R = \mathbb{Z}/8[x] \) and \( f(x) = x^2 - c \) for some \( c \in \mathbb{Z}/8 \).] Where does your proof in part (a) break down in this situation?

3. Which of the following ideals are maximal in the indicated rings? For those that are not, find a maximal ideal containing the given ideal. Explain your assertions. [Caution: One of these is tricky.]
   \( (x - 3) \subset \mathbb{Q}[x] \); \( (x - 3) \subset \mathbb{Z}[x] \); \( (x^2 - 3) \subset \mathbb{R}[x] \); \( (x^2 + 3) \subset \mathbb{R}[x] \); \( (x - 3) \subset \mathbb{C}[x, y] \);
   \( (x^2 + 1, y - 3) \subset \mathbb{R}[x, y] \); \( (x^2 + 1, y - 3) \subset \mathbb{C}[x, y] \); \( (x^2 + 1, y^2 + 1) \subset \mathbb{R}[x, y] \).

Part C:
From Artin, do these problems (at the end of Chapter 10):
Section 10.5: 16; 10.7: 9, 11; Miscellaneous problems: 2.

Also do the following problem:
   a) Show that \( \sqrt{2} \) is irrational.
   b) More generally, show that if \( m \in \mathbb{Z} \) and \( x^2 - m \) has no root in \( \mathbb{Z} \), then \( x^2 - m \) has no root in \( \mathbb{Q} \).
   c) Still more generally, show that if \( a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z} \), and if the polynomial \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \) has no root in \( \mathbb{Z} \), then it has no root in \( \mathbb{Q} \).
   d) What if, in part (c), the polynomial \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) (for some integers \( a_0, a_1, \ldots, a_n \)) is considered instead?