Math 503 Problem Set #4 Due the week of Feb. 5, 2007, in lab.

Read Artin, Chapter 10, sections 7,8; Chapter 11, section 1,2.

Part A:
From Artin, do these problems (given at the end of Chapter 10):
Section 10.7: 6; 10.8: 1(a,c), 4, 5 (is it also true over $\mathbb{R}$?).

From Artin, do these problems (given at the end of Chapter 11):
Section 11.1: 9; 11.2: 1.

Part B:
1. Let $f(x,y) = x^2 + y^2 + 1$.

a) Is the principal ideal $(f) \subset \mathbb{C}[x,y]$ maximal? If not, find a maximal ideal containing it.

b) Do the same with $\mathbb{C}$ replaced by $\mathbb{R}$.

2. a) Which of the following are maximal ideals of $\mathbb{C}[x,y]/(x^2+y^2-25)$? For each that is not, explain why not.
$(x-3, y-4)$; $(x-1, y-1)$; (x); $(x-5)$.

b) Do the same for the ring $\mathbb{C}[x,y]/(y-x^2)$.

3. Find the maximal ideals of $\mathbb{Z}[1/2]$. Do the same for $\mathbb{C}[x, x^{-1}]$.

4. Let S be a ring and let $R \subset S$ be a subring. If I is an ideal of S, define its *contraction* I^c to R to be $I \cap R$.

a) Show that I^c is an ideal of R.

b) If I is a maximal ideal of S, must I^c be a maximal ideal of R? Do this in the following two cases:

i) $R = \mathbb{C}[x]$, $S = \mathbb{C}[x,y]$.

ii) $R = \mathbb{C}[x]$, $S = \mathbb{C}(x)$.

Part C:
From Artin, do these problems (at the end of Chapter 10):
Section 10.7: 12; 10.8: 13 (#13(b) is over $\mathbb{C}$; it is true over $\mathbb{R}$?); Miscellaneous problems: 24(a-c).

From Artin, do this problem (given at the end of Chapter 11):
Section 11.1: 4.

Also do the following problem:

A commutative ring R is called *local* if R has exactly one maximal ideal.

a) Which of the following rings are local? $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}[x]/(x^3)$, $\mathbb{R}[[x]]$, $\mathbb{R}[x]$, $\mathbb{R}(x)$.

b) Let R be the subring of $\mathbb{C}(x)$ consisting of rational functions that are defined at $x = 0$ (and therefore in some neighborhood of that point). Show that R is local. (This example gives rise to the terminology. A function in $\mathbb{C}[x]$ is viewed as "global.")

c) Let R be the subring of $\mathbb{Q}$ consisting of rational numbers with odd denominator. Show that R is local.