Math 503 Problem Set #9 Due the week of March 19, 2007, in lab.

Read Artin, Chapter 12, sections 5-7.

Part A:
From Artin, do these problems (given at the end of Chapter 12):
12.5: 5; 12.6: 1, 2, 4; 12.7: 1, 2 (also find the rational canonical form), 6.

Part B:
1. Determine which of the following rings are Noetherian:
$\mathbb{Z}[i,x,y]/(x^2+iy^3)$; $\mathbb{R}[[x]][y]$; $\mathbb{Z}[\sqrt{2}]$; $\mathbb{Z}[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots]$; $\mathbb{Z}[1/2]$; $\mathbb{Z}[1/2, 1/3, 1/4, \ldots]$.

2. a) In the situation of Problem Set #7, problem 3, suppose that R is a field. Show that if ϕ is surjective then so is ϕ_*. Also show that if ψ is injective then ψ^* is surjective.

b) Show by example that the conclusion of part (a) does not necessarily hold for modules over an arbitrary commutative ring R. [Hint: Take $R = \mathbb{Z}$, let one of the modules be $\mathbb{Z}/2$, and let one of the maps be multiplication by 2.]

3. Let G be a group. For any positive integer n, let G_n consist of the elements of G whose order divides n. Let G_{tor} be the union of all the sets G_n. (So the elements of G_{tor} are the elements of G having finite order; these are also called "torsion elements.")

a) Show that if G is abelian, then G_n and G_{tor} are subgroups of G.

b) Show by example that if G is non-abelian then these subsets need not be subgroups of G. [Hint: For G_{tor}, consider the infinite dihedral group.]

4. Let $G = \prod_{i=1}^{s} \mathbb{Z}/d_i$. In the notation of problem 3, show that G_n is isomorphic to $\prod_{i=1}^{s} \mathbb{Z}/(n, d_i)$, where (n, d_i) is the greatest common divisor of n and d_i.

Part C:
From Artin, do these problems (at the end of Chapter 12):
Section 12.7: 18 (do this using rational canonical form; first try the case of one block); Miscellaneous problems: 7, 11 [Hint: Pick $g \in A$ and make the substitution $a = bg$, where b is a new variable.]

Also do the following problem:

Suppose that $\prod_{i=1}^{s} \mathbb{Z}/d_i$ is isomorphic to $\prod_{i=1}^{s'} \mathbb{Z}/d'_i$, where $d_1|d_2|\cdots|d_s$ and $d'_1|d'_2|\cdots|d'_{s'}$. Prove $s = s'$ and $d_i = d'_i$ for all i. [Hint: Use problem 4 above for suitable choices of n.]