Math 503 Problem Set #10 Due the week of March 26, 2007, in lab.

Read Artin, Chapter 13, sections 1-3.

Part A:
From Artin, do these problems (given at the end of Chapter 12): Miscellaneous problems 8, 9.
From Artin, do these problems (given at the end of Chapter 13): Section 13.1: 1, 4; 13.2: 1; 13.3: 1, 2.

Part B:
1. a) If R is a commutative ring, find an isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \xrightarrow{\sim} R[x]/(x^2-2)$.
   b) Determine whether $\mathbb{Z}[\sqrt{3}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ are integral domains.
   c) Simplify each of the following $\mathbb{Z}$-modules (up to isomorphism):
$\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/6$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Z}/10 \otimes \mathbb{Z}^3$.

2. Let R be a commutative ring, let $\phi : N_1 \to N_2$ be a homomorphism of R-modules, and let M be an R-module.
   a) Show that there is an induced homomorphism of R-modules $\phi_* : M \otimes N_1 \to M \otimes N_2$ defined by $\phi_*(m \otimes n) = m \otimes \phi(n)$.
   b) Show that if ϕ is surjective then so is ϕ_*.
   c) Show that if M is a free R-module (e.g. if R is a field), then if ϕ is injective then so is ϕ_*. But show by example that if M is arbitrary, then it is possible for ϕ to be injective and ϕ_* not to be injective. [Compare PS #9, problem B2.]

3. For each of the following field extensions F of $\mathbb{Q}$, find the degree of F over $\mathbb{Q}$ and find the group $\mathrm{Aut}(F)$ of automorphisms of F.
$\mathbb{Q}$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\zeta_5]$ (where ζ_5 is a primitive fifth root of unity), $\mathbb{Q}[\sqrt[4]{2}]$, $\mathbb{Q}[\sqrt[5]{2}]$

Part C:
From Artin, do these problems (at the end of Chapter 13):
Section 13.1: 2, 3; 13.2: 3.

Also do the following problem:
Let $K = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{2+\sqrt{2}}]$.
a) Find the multiplicative inverse of $\sqrt{2+\sqrt{2}}$ in L (as a polynomial in $\sqrt{2+\sqrt{2}}$).
b) Show $K \subset L$. What is $[K : \mathbb{Q}]$? $[L : K]$? $[L : \mathbb{Q}]$?
c) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_{\mathbb{Q}}$?
d) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_K$?
e) Find an element of order 4 in $\mathrm{Aut}(L)$. What is the group $\mathrm{Aut}(L)$ abstractly?