Math 503 Problem Set #11 Due the week of April 2, 2007, in lab.

Read Artin, Chapter 13, sections 3-5.

Part A:
From Artin, do these problems (given at the end of Chapter 13): Section 13.3: 3(a-d), 6, 10, 12, 15; 13.4: 1; 13.5: 4.

Part B:
1. a) Prove that if a rational number $\alpha = a/b$ (in lowest terms) is a root of a polynomial $c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$ of degree $n > 0$, then $a|c_0$ and $b|c_n$.

b) Find all positive integers r, s such that the polynomial x^3+rx^2+sx+1 is irreducible in $\mathbb{Q}[x]$. [Hint: Use part (a).]

2. a) Given a point P in the plane and a circle C with center P, show that for every positive integer n one can construct (with straightedge and compass) a circle whose circumference is n times that of C, and a circle whose area is n times that of C.

b) What happens in instead one is given a point P in 3-space and a sphere S with center P, and one wants to find spheres whose surface area or volume is n times that of S? [Note: For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $\frac{4}{3}\pi r^3$.]

3. a) Show that if K is a field whose characteristic is not 2, and if $L = K[\sqrt{a}]$ and $L' = K[\sqrt{b}]$ are two field extensions of K of degree 2, then L and L' are isomorphic as field extensions of K if and only if there exists $c \in K$ such that $b = ac^2$.

b) Show that this fails in characteristic 2. [Hint: Take $K = \mathbb{F}_2(t)$.]

Part C:
From Artin, do these problems (at the end of Chapter 13):
Section 13.3: 8, 9; 13.4: 4, 10.

Also do the following problem:

a) Show that $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] \approx \mathbb{Q}[i, \sqrt{2}]$. [Hint: Find a map and compare degrees.]

b) Show that $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$ is not a field, and is isomorphic to $\mathbb{Q}[i] \times \mathbb{Q}[i]$. [Hint: Send $\alpha \otimes \beta$ to $(\alpha\beta, \alpha\bar{\beta})$, where bar indicates complex conjugation.]

c) Make a conjecture about when $\mathbb{Q}[\sqrt{a}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{b}]$ is a field, if a, b are square-free integers; and about what explicit field or ring it is isomorphic to.

[Note: If $R \subset A, B$ for some rings R, A, B, then the R-module $A \otimes_R B$ has a ring structure given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.]