Math 503 Problem Set #13 Due the week of April 16, 2007, in lab.

Reminder: There will be an exam in class on Wed., April 18, on the material beyond what was covered on the first exam (i.e. beyond Section 12.2 of Artin).

Read Artin, Chapter 14, sections 2-7.

Part A:
From Artin, do these problems (given at the end of Chapter 14): Section 14.1: 1, 2, 17; 14.4: 1; 14.5: 9 [You may rely on results proven in the text. Also see problem B1.]; Miscellaneous problems: 4.

Part B:
1. a) Find the degree of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$, and also find its minimal polynomial.
b) Show that α is a primitive element for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
c) Repeat part (a) for $\beta = \sqrt[4]{2}$. Is $\mathbb{Q}(\beta)$ Galois over $\mathbb{Q}$? [Hint: Is it normal?]

2. Let $L = \mathbb{Q}[\sqrt{2+\sqrt{2}}]$.
a) Show that L is Galois over $\mathbb{Q}$, and that its Galois group is cyclic of order 4. [Hint: PS 10, Part C problem.]
b) Show that L is not the splitting field over $\mathbb{Q}$ of a polynomial $x^4 - a$ for some $a \in \mathbb{Q}$.
c) What happens to part (a) if we instead consider the field $\mathbb{Q}[\sqrt{3+\sqrt{3}}]$?

3. Let $K = \mathbb{C}(x)$ and $L = K[\sqrt[6]{x}]$. Show that L is Galois over K, find the Galois group, find all intermediate fields M (i.e. fields with $K \subset M \subset L$), and find all the Galois groups $\mathrm{Gal}(L/M)$ and $\mathrm{Gal}(M/K)$ for these fields M. Verify in this example that M is Galois over K if and only if $\mathrm{Gal}(L/M)$ is a normal subgroup of $\mathrm{Gal}(L/K)$, and that $\mathrm{Gal}(M/K)$ is then the quotient of these two groups.

Part C:
From Artin, do these problems (at the end of Chapter 14):
Section 14.5: 2 [Hint: The Galois group permutes the roots of f.], 12; 14.7: 1 [You may assume $\zeta_n \in F$.].

Also do the following problem:
Redo problem B3 above for the fields $K = \mathbb{Q}$ and $L = K[\zeta_3, \sqrt[3]{2}]$, where ζ_3 is a primitive cube root of unity.