Math 503 Problem Set #1 Due the week of Jan. 15, 2018, in lab.

Read Artin, Chapter 11, section 1.

Part A:

For each of the following sets, determine whether it is a ring, and explain why or why not. (Terminology here: To be a "ring," it must have a multiplicative identity $1 \neq 0$. Assume that the operations are given by addition and multiplication unless stated otherwise.) For those that are rings, say what the identity elements are under the two operations, and say which elements have multiplicative inverses. Also say whether the ring is commutative (under multiplication).

1. $\{a + b\sqrt{7} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.
2. $\{a + b\sqrt[3]{5} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.
3. $M_n(\mathbb{Z}[x])$.
4. $\mathbb{Z} \times \mathbb{R}$, with $(a, b) + (c, d) = (a + c, b + d)$, $(a, b) \cdot (c, d) = (ac, bd)$.
5. $\mathbb{R} \times \mathbb{R}$, with $(a, b) + (c, d) = (a + c, b + d)$, $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.
6. $x\mathbb{R}[x]$, the polynomials that are multiples of x.
7. $\{$continuous functions $f : \mathbb{R} \to \mathbb{R}\}$ under addition and composition.

Part B:

For each of the following maps, determine whether it is a homomorphism of rings, and explain why or why not. For those that are homomorphisms, determine whether they are injective and whether they are surjective.

8. $f : \mathbb{R}[x] \to \mathbb{R}$, $f(\sum a_i x^i) = \sum a_i 3^i \quad (a_i \in \mathbb{R})$.
9. $f : \mathbb{C} \to \mathbb{C}$, $f(a + bi) = a - bi \quad (a, b \in \mathbb{R})$.
10. $f : \mathbb{C} \to \mathbb{C}$, $f(a + bi) = a \quad (a, b \in \mathbb{R})$.
11. $f : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{3}]$, $f(a + b\sqrt{2}) = a + b\sqrt{3} \quad (a, b \in \mathbb{Z})$. (Notation used here: $\mathbb{Z}[\sqrt{n}] = \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\}$.)
12. $f : \mathbb{Z}[x] \to \mathbb{Q}$, $f(\sum a_i x^i) = \sum a_i/2^i \quad (a \in \mathbb{Z})$.
13. $f : \mathbb{Z}[i] \to \mathbb{Z}/n\mathbb{Z}$, $f(a + bi) = a + 8b \quad (a, b \in \mathbb{Z})$. (Hint: Your answer should depend on n. Here $i = \sqrt{-1}$.)
14. $f : \mathbb{C} \to M_2(\mathbb{R})$, $f(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad (a, b \in \mathbb{R})$.