Math 503 Problem Set #9 Due the week of March 19, 2018, in lab.

In Artin, Chapter 14, review sections 1-4 and read sections 5-7.

1. From Artin, Chapter 14, do these problems (pages 437-441): 1.4, 2.4, 4.1(a) (second matrix only), 5.1, 5.2.

2. Let R be a commutative ring.

a) If M, N are R-modules, let $\mathrm{Hom}_R(M, N)$ be the set of R-module homomorphisms $M \to N$. Show that $\mathrm{Hom}_R(M, N)$ is an R-module.

b) Show that if M and N are free R-modules of rank m and n respectively, then $\mathrm{Hom}_R(M, N)$ is free R-module of rank mn. [Hint: Use matrices.]

3. Let R be a commutative ring.

a) Show that if $\phi : N_1 \to N_2$ is a homomorphism of R-modules, and if M is an R-module, then there is a homomorphism of R-modules $\phi_* : \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2)$ defined by $\phi_*(f) = \phi \circ f$.

b) Show that if $\psi : M_1 \to M_2$ is a homomorphism of R-modules, and if N is an R-module, then there is a homomorphism of R-modules $\psi^* : \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ defined by $\psi^*(g) = g \circ \psi$.

4. Let R be a commutative ring.

a) Show that if M is an R-module, then $\mathrm{Hom}_R(R, M)$ is isomorphic to M as an R-module. [Hint: Send f to $f(1)$.]

b) If M is an R-module, define its *dual module* by $M^* = \mathrm{Hom}_R(M, R)$. If $R = \mathbb{Z}$, find M^* in each of these cases: $M = \mathbb{Z},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}/3,\ \mathbb{Q}$. Also, explain the relationship of dual modules to dual vector spaces.

c) Evaluate $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/n)$ in each of these cases: $n = 2, 3, 4$.

5. Consider the ideal $I \subset \mathbb{C}[x, y]$ generated by the set $\{y - x, y - x^2, y - x^3, \ldots\}$. Is I finitely generated? If so, find an explicit finite generating set. Also find the zero locus of I in $\mathbb{C}^2$.