Math 503 Problem Set #10 Due the week of March 26, 2018, in lab.

In Artin, Chapter 14, read sections 6-9.

1. From Artin, Chapter 14, do these problems (pages 437-441): 6.1, 7.1, 7.2, 7.5, 8.1.

2. Which of the following R-modules are finitely generated? Which are free?

a) $R = \mathbb{Z}[\sqrt{-5}]$, $M = (2, 1 + \sqrt{-5}) \subset R$
b) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]$
c) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]/(y^2 - x)$
d) $R = \mathbb{R}[x, y]$, $M = (x, y) \subset R$
e) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, 1/x]$.

3. Determine which of the following rings are Noetherian:
the ring of integers of $K = \mathbb{Q}[\sqrt{d}]$ where d is a square-free integer; $\mathbb{Z}[i, x, y]/(x^2 + iy^3)$; $\mathbb{Z}[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots]$; $\mathbb{Z}[1/2]$; $\mathbb{Z}[1/2, 1/3, 1/4, \ldots]$.

4. Let G be a group. For any positive integer n, let G_n consist of the elements of G whose order divides n. Let G_{tor} be the union of all the sets G_n. (So the elements of G_{tor} are the elements of G having finite order; these are also called "torsion elements.")

a) Show that if G is abelian, then G_n and G_{tor} are subgroups of G.

b) Show by example that if G is non-abelian then these subsets need not be subgroups of G. [Hint: For G_{tor}, consider the infinite dihedral group.]

5. a) Let $G = \prod_{i=1}^{s} \mathbb{Z}/d_i$. In the notation of problem 4, show that G_n is isomorphic to $\prod_{i=1}^{s} \mathbb{Z}/(n, d_i)$, where (n, d_i) is the greatest common divisor of n and d_i.

b) Prove the uniqueness part of the fundamental theorem of finite abelian groups in the following form: If $\prod_{i=1}^{s} \mathbb{Z}/d_i$ is isomorphic to $\prod_{i=1}^{s'} \mathbb{Z}/d'_i$, where $d_1 | d_2 | \cdots | d_s$ and $d'_1 | d'_2 | \cdots | d'_{s'}$, and where $d_i, d'_i \geq 2$, then $s = s'$ and $d_i = d'_i$ for all i. [Hint: Use part (a) for suitable choices of n.]