Math 503 Problem Set #11 Due the week of April 2, 2018, in lab.

In Artin, Chapter 15, read sections 1-4.

1. From Artin, Chapter 15, do these problems (pages 472-476): 1.1, 2.1, 3.1, 3.3, 4.2(a).

2. Consider the situation of Problem Set #9, problem 3.

a) Show that if ϕ is injective then so is ϕ_*. Also show that if ψ is surjective then ψ^* is injective.

b) Suppose that R is a field. Show that if ϕ is surjective then so is ϕ_*. Also show that if ψ is injective then ψ^* is surjective.

c) Show by example that the conclusion of part (b) does not necessarily hold for modules over an arbitrary commutative ring R. [Hint: Take $R = \mathbb{Z}$, let one of the modules be $\mathbb{Z}/2$, and let one of the maps be multiplication by 2.]

3. For each of the following field extensions F of $\mathbb{Q}$, find the degree of F over $\mathbb{Q}$ and find the group $\mathrm{Aut}(F)$ of automorphisms of the field F.
$\mathbb{Q}$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\zeta_5]$ (where ζ_5 is a primitive fifth root of unity), $\mathbb{Q}[\sqrt[4]{2}]$, $\mathbb{Q}[\sqrt[5]{2}]$.

4. Let $K = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{2+\sqrt{2}}]$.

a) Find the multiplicative inverse of $\sqrt{2+\sqrt{2}}$ in L (as a polynomial in $\sqrt{2+\sqrt{2}}$).

b) Show $K \subset L$. What is $[K:\mathbb{Q}]$? $[L:K]$? $[L:\mathbb{Q}]$?

c) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_{\mathbb{Q}}$?

d) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_K$?

e) Find an element of order 4 in $\mathrm{Aut}(L)$. What is the group $\mathrm{Aut}(L)$ abstractly?

5. a) Prove that if a rational number $\alpha = a/b$ (in lowest terms) is a root of a polynomial $c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{Z}[x]$ of degree $n > 0$, then $a|c_0$ and $b|c_n$.

b) Find all positive integers r, s such that the polynomial x^3+rx^2+sx+1 is irreducible in $\mathbb{Q}[x]$. [Hint: Use part (a).]