Quadratic Forms (Math 520/620/702)

Problem Set #2

1. Let $F = \mathbb{C}((t))$.
   a) Show that $F^\times / F^\times 2$ has exactly two elements, represented by \{1, t\}. [Hint: Show that if $f \in \mathbb{C}[[t]]$ has a non-zero constant term, then $f$ is a square.]
   b) Show that every binary quadratic form over $F$ is universal. [Hint: Use part (a) to reduce to just a few possibilities.]
   c) Deduce that $u(F) = 2$.
   d) Describe the structure of $W(F)$, $Q(F)$, $I(F)$, and $I^2(F)$.

2. Let $k$ be an arbitrary field of characteristic unequal to 2. Let $F = k((t))$ and $R = k[[t]]$.
   a) Let $f$ be an element of $R$ with constant term $c$. Show that $f$ is a unit in the ring $R$ if and only if $c \neq 0$. Also show that if $c = 1$ then $f$ is a square in $R$. [Hint: Taylor series for $(1 + x)^{1/2}$.] Deduce that if $c \neq 0$, then $f$ is a square in $R$ (and in $F$) if and only if $c \in k^\times 2$.
   b) Let $q = \langle a_1, \ldots, a_n \rangle$ with $a_i \in R^\times$, the group of units in $R$. Show that if $q$ is isotropic over $F$ then $q(x) = 0$ for some $x = (x_1, \ldots, x_n) \in R^n$ that does not lie in $tR^n$.
   c) In (b), write $a_i = c_{i,0} + c_{i,1} t + c_{i,2} t^2 + \cdots$ with $c_{i,j} \in k$, and write $\bar{q} = \langle c_{1,0}, \ldots, c_{n,0} \rangle$. Show that if $q$ is isotropic over $F$ then $\bar{q}$ is isotropic over $k$. [Hint: Use part (b) and then reduce mod $(t)$.]
   d) Prove the converse of part (c). [Hint: Use part (a).]

3. In this problem, we retain the notation of problem 2.
   a) Show that every regular quadratic form over $F$ is equivalent to a quadratic form $q_1 \perp t q_2$ for some $q_1 = \langle a_1, \ldots, a_r \rangle$ and some $q_2 = \langle a_{r+1}, \ldots, a_n \rangle$, where each $a_i \in R^\times$. Show moreover that if $\bar{q}_1$ or $\bar{q}_2$ is isotropic, then so is $q$. [Hint: Use problem 2(d).]
   b) Prove the converse of the last part of (a). [Hint: First obtain an $x \in R^n$ as in problem 2(b). Next, consider the case in which at least one of the elements $x_1, \ldots, x_r \in R$ has non-zero constant term; and handle this case by modding out by $(t)$. Finally, handle the remaining case by showing that the form $t^2 q_1 + t q_2$ is also isotropic over $R$, and then dividing by $t$ and reducing to the previous case.]
   c) Using parts (a) and (b), find and prove a formula that relates $u(F)$ to $u(k)$.

4. Let $F$ be a finite field whose order is congruent to 3 modulo 4. Show directly that $\langle -1, -1 \rangle = \langle 1, 1 \rangle$ in $\hat{W}(F)$ and hence in $W(F)$; that $\langle 1, 1, 1 \rangle = \langle -1 \rangle$ in $W(F)$; and that $\langle 1, 1, 1, 1 \rangle$ is trivial in $W(F)$. (Here, “directly” means that you should work with these quadratic forms, and not just use that we showed that $W(F)$ is abstractly isomorphic to $\mathbb{Z}/4$.)

5. a) Verify the assertion in Lam (page 34 of the text, and also exercise 18 on page 49) that the subfield $\bigcup_{n \geq 1} \mathbb{F}_5(\sqrt{2})$ of $\overline{\mathbb{F}}_5$ is quadratically closed (and hence is the quadratic closure of $\mathbb{F}_5$ in $\overline{\mathbb{F}}_5$).
   b) Find a similar description for the quadratic closure of $\mathbb{F}_3$ in $\overline{\mathbb{F}}_3$. 

Due Mon., Oct. 24, 2011, in class.
c) Is there a similar simple description for the quadratic closure of $\mathbb{Q}$? Why or why not?

6. Do the following problems from Lam, Chapter II (pages 47-49):
   a) Exercise 2. [Hint: Use another form of equivalence.]
   b) Exercise 5. [Hint: Proceed similarly to Theorem 3.5.]
   c) Exercise 6(a). [Hint: See the example on page 35.]
   d) Exercise 7(a). [Hint: First do the case $n = 2$ (and even $n = 1$), and guess a pattern.]
   e) Exercise 10. [Hint: Reinterpret the condition on $-1$ in terms of isotropy.]
   f) Exercise 11(1). [Hint: Use the explicit group isomorphism $\mathbb{Q}(F) \to W(F)/I^2(F)$ to reinterpret the multiplication on $W(F)/I^2(F)$ as an operation $\circ$ on $\mathbb{Q}(F)$.]