1. Let $A$ be a central simple algebra over $F$, and let $E$ be a field that contains $F$ and is contained in $A$.
   a) Show that the centralizer $C_A(E)$ contains $E$, and is an $E$-algebra.
   b) Show that $\dim_F(C_A(E)) = [E : F]$. [Hint: What is $\dim_F(E) \cdot \dim_F(C_A(E))$?]
   c) Deduce that $[E : F]$ divides the degree of the $F$-algebra $A$, with equality if and only if $C_A(E) = E$. [Hint: What is $\dim_F(E)$?
   d) Show that if $[E : F]$ is equal to the degree of $A$, then $E$ is a maximal subfield of $A$ (i.e. $E$ is not strictly contained in any other field $E'$ with $F \subseteq E' \subseteq A$).
   e) Show that if $A$ is a division algebra over $F$ then the converse of (d) holds. [Hint: If not, show there exists $a \in C_A(E)$ that does not lie in $E$, and consider $E(a) \subseteq A$.]

2. a) Let $D$ be a non-commutative division ring that is also a finite dimensional $\mathbb{R}$-algebra. Show that the center must be $\mathbb{R}$, and hence $D$ is a (central) division algebra over $\mathbb{R}$. [Hint: If not, $D$ is a non-trivial central simple algebra over the field $\mathbb{Z}(D)$. What can that field be?]
   b) Let $E$ be a maximal subfield of the $\mathbb{R}$-division algebra $D$. Show that $E \cong \mathbb{C}$ and that the degree of $D$ over $\mathbb{R}$ is 2. Deduce that $D$ is a quaternion algebra over $\mathbb{R}$.
   c) Conclude that $\text{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

3. Let $K$ and $A$ be as in problem 6 of Problem Set 3, and preserve the notation from that problem.
   a) Show that if $A$ is a $K$-division algebra then $A$ contains a maximal subfield $E$ of degree $n$ over $K$ such that $E$ is a cyclic Galois extension of $K$, i.e. a Galois extension of $K$ whose Galois group is cyclic. (For this reason, $A$ is referred to as a cyclic algebra.) Find the centralizer of $E$ in $A$.
   b) Show that if $b = 1$ then $A$ is isomorphic to a matrix algebra over $K$. [Hint: Consider the matrices $M, N$.] What does this say if $n = 2$?
   c) Given an example to show that $A$ is not always isomorphic to a matrix algebra.

4. If $\sigma$ is a permutation of $\{1, 2, 3, 4\}$, consider the map $f_\sigma : \mathbb{H} \to \mathbb{H}$ that takes $a_1 + a_2i + a_3j + a_4k$ to $a_{\sigma(1)}i + a_{\sigma(2)}j + a_{\sigma(3)}j + a_{\sigma(4)}k$, where each $a_i \in \mathbb{R}$.
   a) For which permutations $\sigma$ is $f_\sigma$ an automorphism of $\mathbb{H}$?
   b) Concerning each such $\sigma$, what assertion does the Skolem-Noether Theorem make?
   c) Verify this assertion explicitly by finding an element as asserted in that theorem, for one such choice of $\sigma$ (other than the identity).

5. Do the following problems from Lam, Chapter V (pages 140-142):
   a) Exercise 4.
   b) Exercise 12.
   c) Exercise 14.