1. Let $G$ be a $p$-group, and let $\Phi$ be its Frattini subgroup.
   a) Show that if $g \in G$ then $g^p \in \Phi$. (Hint: If $H \subset G$ is a maximal subgroup, show that $g^p \in H$ by considering its image in $G/H$.)
   b) Deduce that every element of $G/\Phi$ has order 1 or $p$.
   c) Conclude that $G/\Phi$ is isomorphic to $(\mathbb{Z}/p)^n = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ (with $n$ factors) for some $n \geq 0$.

2. If $K$ is a group and $S$ is a subset of $K$ that generates $K$, we will call $S$ a minimal generating set for $K$ if no proper subset of $S$ also generates $K$.
   a) Show that every minimal generating set of $(\mathbb{Z}/p)^n$ has exactly $n$ elements. (Hint: View $(\mathbb{Z}/p)^n$ as a vector space.)
   b) Prove or disprove: If $G$ is any finite group, then any two minimal generating sets for $G$ have the same number of elements.
   c) Let $G$ be a $p$-group with Frattini subgroup $\Phi$, so that $G/\Phi$ is isomorphic to $(\mathbb{Z}/p)^n$ (as in problem 1(c)). Show that
      (i) Every minimal generating set for $G$ has exactly $n$ elements.
      (ii) If $T$ is a subset of $G$ with exactly $n$ elements, then $T$ is a minimal generating set for $G$ if and only if its image under $G \rightarrow G/\Phi$ is a minimal generating set for $G/\Phi$.
   (Hint: Use Problem Set 2 #6 and problems 1(c) and 2(a) above.)
   (Remark: Part (c) is also called the Burnside Basis Theorem.)

3. For each $n$, $9 \leq n \leq 16$, answer the following questions:
   a) Is every group of order $n$ cyclic?
   b) Is every group of order $n$ abelian?
   c) Is every abelian group of order $n$ cyclic?
   Justify your answers.

4. a) Show that every element of $A_5$ is conjugate (in $A_5$) to exactly one of the following five elements:
      $$1, (123), (12)(34), (12345), (12354).$$
   Determine the number of elements conjugate to each.
   b) Deduce that $A_5$ is simple. [Hint: Show that every normal subgroup is a union of conjugacy classes. Then apply part (a) and Lagrange’s Theorem.]

5. Suppose that $N \triangleleft S_5$.
   a) Show that if $N$ contains a transposition $(a,b)$ then $N = S_5$. (Hint: The set of transpositions generates $S_5$.)
   b) Show that if $N \cap A_5 = 1$ and $\sigma \in N$, then either $\sigma = 1$ or else $\sigma$ is a transposition.
   (Hint: Show that $\sigma^2 = 1$.)
   c) Conclude that $N = 1, A_5, or S_5$. [Hint: Apply 5(b) to $N \cap A_5$, and use parts (a) and (b) above.]

6. Show that the three definitions of “solvable” are equivalent: that there is a sequence of subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with each $G_i/G_{i-1}$ abelian (respectively, cyclic, or cyclic of prime order).