1. a) If $A$ is a square matrix satisfying $A^3 - A^2 + A - I = 0$, find $A^{-1}$ in terms of $A$.
   b) If $A$ is a square matrix satisfying $A^7 = 0$, find $(I - A)^{-1}$ in terms of $A$.

2. a) Let $V = \{ \text{differentiable functions on } \mathbb{R} \}$. Prove that the functions $e^x, e^{2x}, e^{3x}$ are linearly independent in the vector space $V$. [Hint: If not, differentiate twice.]
   b) Let $W$ be the set of solutions to the differential equation $f''' - f = 0$, and let $V$ be the set of solutions to $f''' - f' = 0$. Show that $W$ is a vector subspace of $V$, find a basis for $W$, and extend this basis to a basis of $V$.

3. Let $n$ be an integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $P_n \subset \mathbb{R}[x]$ be the vector space of polynomials of degree $\leq n$. Define $F : P_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.
   a) Show that $F$ is an isomorphism. [Hint: $\dim P_n = \ker F$?]
   b) Explicitly find $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ (where $e_1, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{R}^{n+1}$) in the case $n = 3$, $\alpha_j = j (j = 1, 2, 3, 4)$. [Hint: where does $(x-a)(x-b)(x-c)$ vanish?]
   c) Deduce that $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ form a basis of $P_n$. In the case considered in (b), express $x$ as a linear combination of them.

4. a) If $V$ and $W$ are vector spaces over a field $K$, and if $F : V \to W$ is a homomorphism, let $F^* : W^* \to V^*$ be the map on dual spaces given by $F^*(\phi) = \phi \circ F$. Show that $F \mapsto F^*$ defines a homomorphism $\text{Hom}(V, W) \to \text{Hom}(W^*, V^*)$, satisfying $(F \circ G)^* = G^* \circ F^*$ if $F : V \to W$, $G : U \to V$.
   b) Show that the above map $\text{Hom}(V, W) \to \text{Hom}(W^*, V^*)$ is an isomorphism if $V$ and $W$ are finite dimensional.
   c) Show that if $0 \to U \to V \to W \to 0$ is exact, then so is $0 \to W^* \to V^* \to U^* \to 0$.
   d) What if instead we consider modules over a ring $R$?

5. For any finite dimensional vector space $V$ with basis $B = \{ e_1, \ldots, e_n \}$, and dual basis $B^* = \{ \delta_1, \ldots, \delta_n \}$ of $V^*$, define $\phi_{V, B} : V \to V^*$ by $\sum_i a_i e_i \mapsto \sum_i a_i \delta_i$, and let $\psi_{V, B} = \phi_{V^*, B} \circ \phi_{V, B}$.
   a) Show that $\phi_{V, B} : V \to V^*$ is an isomorphism, but that it depends on the choice of $B$.
   b) Show that $\psi_{V, B} : V \to V^{**}$ is an isomorphism which is independent of the choice of $B$ (so we may denote it by $\psi_V$). For $v \in V$, show that $\psi_V(v)$ is the element of $V^{**}$ taking $f \in V^*$ to $f(v)$.
   c) Show that if $F : V \to W$ is a vector space homomorphism with induced homomorphism $F^{**} : V^{**} \to W^{**}$ (notation as in problem 4), then $\psi_W \circ F = F^{**} \circ \psi_V$.

6. If $U$ is a subspace of a vector space $V$, then the annihilator of $U$ is defined to be $\text{Ann} U = \{ f \in V^* \mid f|_U = 0 \}$.
   a) Show that $\text{Ann} U$ is a subspace of $V^*$. When can it be all of $V^*$? When can it be 0?
   b) Let $V$ be a finite dimensional vector space and let $U, W$ be subspaces of $V$. If $V = U \times W$ (internal direct product), show that $V^* = \text{Ann} U \times \text{Ann} W$.

7. Call $T \in \text{End}(V)$ an idempotent if $T^2 = T$. Show that if $V$ is finite dimensional and $T$ is an idempotent, then there are subspaces $X, Y \subset V$ such that $V = X \times Y$, $T|_X = 0$, $T|_Y = \text{identity}$. Deduce that with respect to some basis of $V$, the idempotent map $T$ is given by a diagonal matrix whose diagonal entries are of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$. 
