1. Define the **center** of a group $G$ to be $Z = \{ g \in G \mid (\forall h \in G) gh = hg \}$.
   a) Is $Z$ a subgroup? Is it normal?
   b) Find the center of $C_n, D_n, S_n, A_n, Q, Z, GL_2(\mathbb{R})$.

2. If $H$ is a subgroup of $G$, define the **normalizer** of $H$ by $N(H) = \{ a \in G \mid aHa^{-1} = H \}$. Is $N(H)$ a subgroup of $G$? Is $H$ a subgroup of $N(H)$? Is $H \triangleleft N(H)$? Is $N(H) \triangleleft G$?

3. a) If $H$ is a subgroup of $G$ and $H \neq G$, we say that $H$ is a **maximal** subgroup if the only subgroups containing $H$ are itself and $G$. Show that if $H$ is maximal then so is $aHa^{-1}$, for any $a \in G$.
   b) Define the **Frattini** subgroup $\Phi$ of $G$ to be the intersection of the maximal subgroups of $G$. Show that $\Phi \triangleleft G$.
   c) Find the Frattini subgroup $\Phi$ of $D_4, C_4, \text{ and } Q$. In each case, find $G/\Phi$. Conjecture?

4. a) If $x \in G$, define its **centralizer** $Z(x) = \{ g \in G \mid xg = gx \}$. Show that $Z(x)$ is a subgroup of $G$, and that its index $(G : Z(x))$ equals the number of elements in the conjugacy class $\{gxg^{-1} \mid g \in G \}$ of $x$.
   b) Consider the conjugacy classes in $G$ that have more than one element. Choose one element from each such class, and gather them together as a set $S$. Show that $|G| = |Z| + \sum_{x \in S} (G : Z(x))$, where $Z$ is the center of $G$.

5. Let $H$ and $K$ be subgroups of $G$. If $k \in K$, call the subgroup $kHk^{-1}$ a **$K$-conjugate** of $H$. Show that the number of $K$-conjugates of $H$ is $(K : K \cap N(H))$, where $N(H)$ is the normalizer of $H$.

6. Define the **commutator** subgroup $G'$ of $G$ to be the subgroup of $G$ generated by the set $C := \{ aba^{-1}b^{-1} \mid a, b \in G \}$.
   a) Show $G' \triangleleft G$.
   b) Show that $G/G'$ is abelian.
   c) Find $G'$ and $G/G'$ if $G = Z, D_4, S_3, C_2 \times C_3$.
   * d) Is it always the case that $G' = C$ for an arbitrary group? for a finite group?

7. a) Show that $\text{Inn } G \triangleleft \text{Aut } G$.
   b) Find $\text{Inn } G$ and $\text{Aut } G$ for $G = S_n, n \leq 4$. Conjecture?

8. Prove or disprove: A group is abelian if and only if every subgroup is normal.