1. a) Let $V$ be an affine variety, with ring of functions $R$. Let $W$ be a Zariski closed subset of $V$, and let $I = I(W)$. Show that $W$ is irreducible if and only if $I$ is a prime ideal.
   b) Let $I_1, \ldots, I_n$ be proper ideals of $R$, and let $W_i = V(I_i)$ for each $i$. Show that
   \[ V(I_1 + \cdots + I_n) = W_1 \cap \cdots \cap W_n, \]
   \[ V(I_1 \cap \cdots \cap I_n) = W_1 \cup \cdots \cup W_n. \]
   Also explain the relationship with problem 1 above.

2. a) Let $R$ be a Noetherian ring and $I \subset R$ an ideal. Prove that there are only finitely many prime ideals that are minimal over $I$. [Hint: If not, show that there is a maximal counterexample $I$, and that this $I$ is not prime. Show that if $a, b \in R - I$ with $ab \in I$, then every prime that is minimal over $I$ is also minimal over either $I + (a)$ or $I + (b)$.] b) Deduce that every Noetherian ring has finitely many minimal primes. Also, interpret this assertion geometrically, if $R$ is the ring of functions on a Zariski closed subset of an affine variety $V$.
   c) What happens in (a) and (b) if the ring is not Noetherian?

3. Determine the Krull dimensions of the following rings: $\mathbb{R}[x, x^{-1}]$, $\mathbb{C}[x, y, z]/(z^2 - xy)$, $\mathbb{Z}[x, y]/(y^2 - x^3)$, $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, $\mathbb{Q}[x, y, z]/(y^2, z^3)$, $\mathbb{Q}[[x, y, z]]$, $\mathbb{Z}[2][x]$. Justify your assertions.

4. Given a commutative ring $R$, define the maximal spectrum of $R$ (denoted $\text{Max} R$) to be the set of maximal ideals of $R$. For each subset $E \subset R$, let $V(E)$ denote the set \{ $m \in \text{Max} R \mid E \subset m$ \} $ \subset \text{Max} R$.
   a) Show that $\text{Max} R$ has a topology in which the closed sets are precisely the sets $V(E)$.
   b) Show that $V(E) = V(I)$ for any $E \subset R$, where $I$ is the ideal generated by $E$.
   c) Show that $V(I) = V(\sqrt{I})$ for any ideal $I$. (Recall that $\sqrt{I}$ denotes the radical of $I$, which is defined to be \{ $r \in R \mid (\exists n) \ r^n \in I$ \}.)
   d) Show that $V(\bigcup_{\alpha} E_\alpha) = \bigcap_{\alpha} V(E_\alpha)$ for any collection of subsets \{ $E_\alpha \}_{\alpha \in A}$, and that $V(I_1 + \cdots + I_n) = V(I_1) \cap \cdots \cap V(I_n)$ for any ideals $I_1, \ldots, I_n$.
   e) Show that $V(I_1 \cap \cdots \cap I_n) = V(I_1) \cup \cdots \cup V(I_n)$ for any ideals $I_1, \ldots, I_n$ of $R$. Also explain the relationship with problem 1 above.
   f) Give examples to illustrate (b) - (e) geometrically, in the case $R = \mathbb{R}[x]$, and in the case $R = \mathbb{Z}$.
   g) If $R = \mathbb{C}[x, y]/(f)$, is there a continuous bijective map between $\text{Max} R$ and the locus of zeroes of $f$ in $\mathbb{C}^2$ (under the usual topology)? In which direction?

5. Consider the following rings: $\mathbb{C}[x]$, $\mathbb{C}[x, y]$, $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$, $\mathbb{C}[x, y]/(x^2 - y^2)$, $\mathbb{C}[x]/(x^2)$, $\mathbb{C}[x, y]/(x^2)$, $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $\mathbb{C}[x]/(x^2 - x)$, $\mathbb{Z}/2$, $\mathbb{Z}/6$, $\mathbb{Z}$, $\mathbb{Z}[1/15]$. For each of them, do the following:
   a) Describe all the maximal ideals in the given ring $R$, and describe $\text{Max} R$ geometrically (or topologically).
b) Determine whether Max $R$ is connected (in the topology given in problem 4).

6. a) Let $R = \mathbb{C}[x, y]/(x^2 - y^2)$ and $S = \mathbb{C}[x, y]/(x^2 - x)$. Show that there is a homomorphism $f : R \to S$ given by $f(x) = y - 2xy$, $f(y) = y$. Show that there is an induced continuous map $f^* : \text{Max } S \to \text{Max } R$ given by $m \mapsto f^{-1}(m)$. Describe the map $f^*$ geometrically. Is it injective? surjective? (A picture in the $(x, y)$-plane may help.)

b) In general, if $f : R \to S$ is a homomorphism of commutative rings, is there an induced continuous map $f^* : \text{Max } S \to \text{Max } R$? (What if $R = \mathbb{Z}$ and $S = \mathbb{Q}$?) What if we instead considered the prime spectrum of $R$ and of $S$? (The prime spectrum $\text{Spec } R$ is defined as the set of prime ideals of $R$ with the topology defined similarly to that of Max.)