1. Which of the following rings $R$ are discrete valuation rings? For those that are, find
the fraction field $K = \text{frac} R$, the residue field $k = R/\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal),
and a uniformizer $\pi$. For the others, explain why not (full proofs not required).
$\mathbb{Z}$, $\mathbb{Z}_{(5)}$, $\mathbb{Z}[1/5]$, $\mathbb{R}[x]$, $\mathbb{R}[x]/(x-2)$, $\mathbb{R}[x,1/(x-2)]$, $\mathbb{Q}[x]/(x^2+1)$,
$\mathbb{C}[x,y]$, $\mathbb{Q}[x,y]/(x^2+y^2-1)(x-1,y)$,
$(\mathbb{R}[x,y]/(y^2 - x^3))(x,y)$.

2. Let $R$ be a discrete valuation ring with fraction field $K$, maximal ideal $\mathfrak{m}$, and discrete
valuation $v$. If $a, b \in K$ define $\rho(a, b) = 2^{-v(a-b)}$ if $a \neq b$, and define $\rho(a, a) = 0$.
   a) Show that $\rho$ defines a metric on $K$.
   b) Show that $\rho$ is an ultrametric (non-archimidean metric); i.e. it satisfies the strong
   triangle inequality $\rho(a, c) \leq \max(\rho(a, b), \rho(b, c))$.
   c) Show that $(K, \rho)$ is a topological field, i.e. that it is a topological space in which
   addition and multiplication define continuous maps $K \times K \to K$.
   d) Show that in $K$, the closed unit disc about 0 is $R$ and the open unit disc about 0
   is $\mathfrak{m}$.

3. Let $K$ be a field and let $f(x) \in K[x]$ be a non-zero polynomial of degree $n$.
   a) Show that if $a \in K$ is a root of $f$, then $(x - a)$ divides $f(x)$ in $K[x]$. [Hint: Use
   the division algorithm for polynomials.]
   b) Deduce that $f$ has at most $n$ roots in $K$.
   c) Will the argument and conclusion of part (b) still hold if $K$ is replaced by a division
   algebra (i.e. if $K$ is no longer assumed commutative)? Explain. [Hint: Try an example.]

4. Let $R$ be a commutative ring of characteristic $p$ (where $p$ is prime) and define $F : R \to R$
by $a \mapsto a^p$.
   a) Show that $F$ is a ring endomorphism (i.e. homomorphism from $R$ to itself).
   b) If $R$ is a field, determine which elements lie in the set $\{a \in R | F(a) = a\}$.
   c) If $R$ is a field, must $F$ be injective? surjective? (Give a proof or counterexample
   for each.)
   d) If $R$ is a finite field, show that $F$ is an automorphism.

5. Let $K$ be a field and let $G$ be a subgroup of the multiplicative group $K^* = K - \{0\}$.
   a) Show that if $a, b \in K$ have finite orders $m, n$, then there is a $c \in K$ whose order is
   the least common multiple of $m, n$. [Hint: First do the case of $m, n$ relatively prime.]
   b) Show that if $G$ is finite then it is cyclic. [Hint: Let $\ell$ be the l.c.m. of the orders
   of the elements of $G$, and apply problem 3(b) to the polynomial $x^\ell - 1$.]
   c) Conclude that if $K \subset L$ is an extension of finite fields, then $L = K[a]$ for some
   $a \in K$. [Hint: What is the group structure of $L^*$?]

The remaining problems are optional, and preserve the notation of problem 2 above.

6. Show that the following conditions are equivalent:
(i) $(R, \rho)$ is a complete metric space.
(ii) $(K, \rho)$ is a complete metric space.
(iii) $R$ is a complete local ring, i.e. $R = \varprojlim R/\mathfrak{m}^n$. 

7. Is $K$ compact if $R = \mathbb{F}_p[[x]]$? If $R = \mathbb{F}_p[x]_x$? If $R = \mathbb{Q}[[x]]$? If $R = \mathbb{Z}_p$ (the $p$-adic integers)? If $R = \mathbb{Z}_p((p))$?

8. a) Show that if $a_1, a_2, a_3, \ldots \in K$ and if $\sum_{n=1}^{\infty} a_n$ converges to an element of $K$, then $\lim_{n \to \infty} a_n = 0$.

b) For which of the rings in problem 7 does the converse to part (a) hold? Can you state and prove a necessary and sufficient condition on $R$ for the converse to hold? Compare and contrast this to the situation for the fields $\mathbb{R}$ and $\mathbb{C}$ under their usual topologies.

9. a) Show that if $f \in K[x]$, then the function $K \to K$ given by $f$ is identically 0 if and only if $f$ is the zero polynomial. Is this true for fields in general?

b) If $f : K \to K$ is a function, define its derivative $f' : K \to K$ by the usual expression $f'(a) = \lim_{h \to 0} (f(a + h) - f(a))/h$, if this exists for all $a \in K$. Show that if $f$ is given by a polynomial in $K[x]$ then its derivative exists, and compute it. Also, find all polynomial functions $f$ such that $f'$ is the zero function. (Your answer should depend on $K$.)