1. Let $p$ be a prime number.
   a) Use Eisenstein’s Irreducibility Criterion to show that the polynomial
   \[ f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 \]
   is irreducible over $\mathbb{Q}$. [Hint: First set $y = x - 1$.]
   b) Give another proof of the same assertion, by first showing that $f(x) = \Phi_p(x)$ (in the notation of Problem Set 9 #2).

2. Under the notation of Problem Set 9 #2:
   a) Describe $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ in terms of $n$. In particular, what is this Galois group when $n = 5, 6, 7, 8, 12$?
   b) For which $n$ is this extension abelian? cyclic? of order 2? of order 3? For which $n$ does it have a cyclic quotient of order 3?
   c) Let $K_7^+ = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$. Find $[K_7 : \mathbb{Q}], [K_7 : K_7^+], \text{and } [K_7^+ : \mathbb{Q}]$. Also find $\text{Gal}(K_7/K_7^+)$ and $\text{Gal}(K_7^+/\mathbb{Q})$.
   d) Find a Galois extension of $\mathbb{Q}$ having degree 5. Find another of degree 7. [Hint: See part (c).]

3. Find the Galois group of (the splitting field of) each of the following polynomials.
   a) $x^3 - 10$ over $\mathbb{Q}$.
   b) $x^3 - 10$ over $\mathbb{Q}(\sqrt{2})$.
   c) $x^3 - 10$ over $\mathbb{Q}(\sqrt{-3})$.
   d) $x^4 - 5$ over $\mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(i)$.
   e) $x^4 - t$ over $\mathbb{R}(t), \mathbb{C}(t)$.

4. Show that for every finite group $G$, there are field extension $\mathbb{Q} \subset K \subset L$ such that $L$ is a finite Galois extension of $K$ with $\text{Gal}(L/K) = G$. (Remark: $[K : \mathbb{Q}]$ is allowed to be infinite.) [Hint: First show the result for $G = S_n$, using Problem Set 9 #3.]

5. a) Find a Galois extension of $\mathbb{Q}$ with Galois group $C_6 \times C_{15}$.
   b) Do the same over the field $\mathbb{F}_5(t)$ (where $\mathbb{F}_5$ is the algebraic closure of $\mathbb{F}_5$).
   c) Let $L = \mathbb{C}(x, y), M = \mathbb{C}(x^2, xy, y^2) \subset L$, and $K = \mathbb{C}(x^2, y^2) \subset M$. Find $[L : M], [M : K], [L : K]$. Is $L$ Galois over $M$? Is $M$ Galois over $K$? Is $L$ Galois over $K$? For those extensions that are Galois, find the Galois group.

6. Let $K$ and $L$ be finite extensions of a field $k$, and let $KL$ be their compositum (inside some fixed algebraic closure).
   a) Find a surjective $k$-algebra homomorphism $\pi : K \otimes_k L \to KL$.
   b) Suppose that $K$ is Galois over $k$. Show that $\pi$ is an isomorphism if only if $K \cap L = k$.
   [Hint: Find $\dim_k(K \otimes_k L)$ and $\dim_k(KL)$,]
   c) Does (b) still hold if $K$ is no longer assumed Galois over $k$?

   The following problem is optional.

7. Hilbert proved the following Irreducibility Theorem: Let $s_1, \ldots, s_m, x_1, \ldots, x_n$ be transcendental over $\mathbb{Q}$. If $f(s_1, \ldots, s_m, x_1, \ldots, x_n)$ is an irreducible polynomial over $\mathbb{Q}$, then
there exist \( \alpha_1, \ldots, \alpha_m \in \mathbb{Q} \) such that \( f(\alpha_1, \ldots, \alpha_m, x_1, \ldots, x_n) \) is an irreducible polynomial in \( \mathbb{Q}[x_1, \ldots, x_n] \). Moreover, if a non-zero polynomial \( g \in \mathbb{Q}[s_1, \ldots, s_m] \) is given in advance, then the \( \alpha \)'s can be chosen so that \( g(\alpha_1, \ldots, \alpha_m) \neq 0 \).

a) Verify this explicitly in the case that \( m = n = 1, f(s, x) = x^3 - s \). Which values of \( \alpha \) work?

b) Show that if \( \mathbb{Q}(s_1, \ldots, s_m) \subset L \) is a finite field extension, then there is an irreducible polynomial \( F \in \mathbb{Q}[s_1, \ldots, s_m, x] \) such that \( L \) is the fraction field of \( \mathbb{Q}[s_1, \ldots, s_m, x]/(F) \).

c) Use Hilbert’s theorem to show that there is an irreducible polynomial \( f(x) \in \mathbb{Q}[x] \) such that the extension \( \mathbb{Q} \subset \mathbb{Q}[x]/(f) \) is Galois with group \( S_n \). [Hint: Part (b) and Problem Set 9 #3.] (Caution: You’ll need to verify that \( \deg(f) = n! \).)

d) Still assuming Hilbert’s theorem, conclude that the field \( K \) in problem 4 above can be chosen to be finite over \( \mathbb{Q} \). [Hint: Reduce to the case of part (c) above.]