1. Let $R$ be a commutative ring, and let $M, N, S$ be $R$-modules. Assume that $M$ is finitely presented and that $S$ is flat. Consider the natural map
\[ \alpha : S \otimes_R \text{Hom}(M, N) \to \text{Hom}(M, S \otimes_R N) \]

taking $s \otimes \phi$ (for $s \in S$ and $\phi \in \text{Hom}(M, N)$) to the homomorphism $m \mapsto s \otimes \phi(m)$.

a) Show that if $M$ is a free $R$-module then $\alpha$ is an isomorphism. [Hint: If $M = R^n$, show that both sides are just $(S \otimes_R N)^n$.]

b) Suppose more generally that $R^a \to R^b \to M \to 0$ is a finite presentation for $M$. Show that the induced diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & S \otimes_R \text{Hom}(M, N) & \longrightarrow & S \otimes_R \text{Hom}(R^b, N) & \longrightarrow & S \otimes_R \text{Hom}(R^a, N) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}(M, S \otimes_R N) & \longrightarrow & \text{Hom}(R^b, S \otimes_R N) & \longrightarrow & \text{Hom}(R^a, S \otimes_R N) & \longrightarrow & 0
\end{array}
\]

is commutative and has exact rows.

c) Using the Five Lemma and part (a), deduce that $\alpha$ is an isomorphism.

2. Let $M$ and $N$ be finitely generated modules over a commutative ring $R$, such that $M \otimes_R N = 0$.

a) Show that if $R$ is a local ring with maximal ideal $m$, then $M$ or $N$ is 0. [Hint: Nakayama’s Lemma.]

b) What if $R$ is not local?

3. Let $M$ be an $R$-module and let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of $R$-modules. Under each of the following four conditions (considered separately), either show that the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ must be exact or else give a counterexample.

a) $M$ is flat.

b) $N'$ is flat.

c) $N$ is flat.

d) $N''$ is flat. [This one is harder, so for now just a conjecture would suffice.]

4. Let $R$ be a commutative ring with Jacobson radical $J$. For $x \in R$, let $P_x$ be the set of $r \in R$ such that $r \equiv 1 \pmod{x}$ (i.e. such that $r - 1 \in xR$). Let $R^x$ denote the multiplicative group of units in $R$.

a) Show that $J = \{x \in R \mid P_x \subseteq R^x\}$.

b) Let $M$ be a finitely generated $R$-module and let $a_1, \ldots, a_n \in M$, where $n \geq 0$. Show that the $R$-module $M$ is generated by $a_1, \ldots, a_n$ if and only if the $R/J$-module $M/JM$ is generated by the images of these elements. [Hint: Either generalize the proof of the local case, or else reduce to that case.]

c) Deduce that if $M$ is a finitely generated $R$-module and $M/JM = 0$ then $M = 0$. Explain why this generalizes Nakayama’s Lemma.