Math 624 Problem Set #4 Due Wednesday, Oct. 15, 2014

Part I:

Read Fulton, Chapter 2, sections 5-11, and Chapter 3, section 1.
Do problems:

2.24, 2.25, 2.31 in Section 2.5;
2.46 in Section 2.8;
2.51 in Section 2.10;
2.56 in Section 2.11;
3.1, 3.3, 3.11 in Section 3.1.

Part II:

Read Hartshorne, Chapter I, section 5.
1. In Chapter I, do problems:
3.17(a,b), 3.21, 4.6, 4.9, 5.1-5.4.

2. Prove that the blow-up of $\mathbb{A}^n$ at the origin is irreducible.

3. (a) Find a blow-up, or series of blow-ups, that makes the curve $y^2 = x^3$ smooth. Describe the proper transform and the total transform.

(b) Do the same for $y^2 = x^5$.

4. Let $P = (0,0) \in \mathbb{A}^2$, $L = (y = 0) \subset \mathbb{A}^2$ over $\mathbb{C}$. Consider the following varieties in $\mathbb{A}^2$:
(i) $y = x^2$; (ii) $y^2 = x^3 - x$; (iii) $y^3 = x^3 - 2x^2$; (iv) $y^2 = x^5$.

For each of these varieties V, do the following:

(a) Draw the (real points of the) graph of V, and find the singular locus of V.

(b) Find the tangent cone to V at P. Also find the tangent lines to V at P and their multiplicities. [If $I(V) = (f)$ and the initial form (of lowest degree terms) of f is $\prod \ell_i^{n_i}$ where the ℓ_i are linear polynomials defining distinct lines L_i, then the L_i are the *tangent lines* at the origin and the *multiplicity* of L_i is n_i. The *tangent cone* is the union of the tangent lines, counting them with their multiplicities.]

(c) Find the tangent space to V at P in $\mathbb{A}^2$ (i.e. the space defined by the linear terms of the above polynomial f). Also calculate $\dim_{\mathbb{C}}(\mathfrak{m}_P/\mathfrak{m}_P^2)$ directly, where $\mathfrak{m}_P$ is the maximal ideal associated to P. Compare. (Also compare to the space spanned by the tangent lines.)

(d) Find the intersection multiplicity $(L \cdot V)_P$ of L with V at P, and find $(L \cdot V)$. [Cf. Hartshorne, Exercise I 5.4.] Does this agree with the real picture?

5. For each of the varieties V in problem 3, and with $P = (0,0)$:

(a) Determine whether $\mathcal{O}_{V,P}$, $\mathcal{O}_{V,P}^{\text{hol}}$, and $\hat{\mathcal{O}}_{V,P}$ are integral domains.

(b) Determine the multiplicity $\mu_P(V)$ of P on V. (See Hartshone, problem I.5.3 for the definition.)

(c) Determine all the nodes of V (i.e. double points with two distinct tangent lines), and all the cusps of V (i.e. points with only one tangent line, which has multiplicity 2).