Math 625 Problem Set #5 Due Monday, March 23, 2015

Read Fulton, Chapter 8, section 6.

Read Hartshorne, Chapter IV, section 6; and Chapter III, sections 1-3.

1. In Fulton, Chapter 8, section 6 do problems 8.31 and 8.34. Optional: problem 8.35.

2. a) In Hartshorne, Chapter IV, do problem 4.8. Optional: problems 4.6, 4.22, 5.1, 5.2, 6.1.

   b) In Hartshorne, Chapter III, do problems 2.1(a), 2.2. Optional: problem 2.1(b).

3. Let X be a smooth connected projective curve of genus g. For any positive integer n, say that X has property D_n if for every choice of distinct points $P_1, \ldots, P_n \in X$, there is a rational function on X with poles of order exactly 1 at each P_i, and no other poles. What can you say about the set of n's such that X satisfies property D_n?

4. Suppose that $\pi : Y \to X$ is an étale morphism; i.e. it is locally given on the ring level by adjoining elements $y_1, \ldots, y_m$ subject to relations $f_1, \ldots, f_m$ such that the relative Jacobian matrix $(\partial f / \partial y)$ is invertible.

   a) Show that if X is a smooth k-scheme then so is Y.

   b) Let $\phi : Z \to X$ be any morphism, and consider the fibre product $Z \times_X Y = \{(z, y) \mid \phi(z) = \pi(y)\}$. Show that the first projection $p_1 : Z \times_X Y \to Z$ is étale. (Hint: First show that p_1 is defined by the same equations as define π.)

5. Suppose that $\pi : Y \to X$ is a finite étale cover of smooth connected k-curves. Let ϕ be an automorphism of the cover, and suppose that $\phi(Q_0) = Q_0$ for some $Q_0 \in Y$.

   a) Show that $Y \times_X Y$ is a smooth curve, and that the first projection $p_1 : Y \times_X Y \to Y$ is étale. (Hint: #4.)

   b) Define $s, t : Y \to Y \times_X Y$ by $s(Q) = (Q, Q)$ and $t(Q) = (Q, \phi(Q))$. Show that s and t are each sections of p_1, and that each is an isomorphism onto its image, with inverse equal to p_1.

   c) Show that the images of s and of t are closed connected subvarieties of $Y \times_X Y$; are of dimension 1; and intersect. Deduce that these images are equal.

   d) Conclude that ϕ is the identity.

   e) In the case of $k = \mathbb{C}$, why is the conclusion of (d) expected?