Math 625 Problem Set #8 Due Wednesday, April 29, 2015

Read Hartshorne, Chapter III, sect. 12; Chapter V, sect. 1-6; and Appendices A and C.

1. In Hartshorne, Chapter III, do problems 11.2 and 11.3 (optional: 11.4 and 12.2). In Chapter V, do problems 1.5 and 2.1 (optional: 1.7, 1.12, 2.6, 3.1, 4.5).

2. Suppose that $f : Y \to X$ is a birational morphism of smooth projective varieties, and let $H \subset Y$ be a hypersurface such that $\dim(f(H)) < \dim(H)$. Prove that H is not linearly equivalent to any effective divisor on Y that meets H properly. [Hint: Otherwise, consider the corresponding rational function on Y, and view it as a rational function on X. What is its divisor there?]

3. What does Riemann-Roch say about the dimension of the space of rational functions on $\mathbb{P}^2$ which have poles at worst D, where D is a given effective divisor of degree d? How can this conclusion also be seen without Riemann-Roch?

4. a) Find an example of a smooth projective surface X, an ample divisor H on X, and a divisor D on X, such that $D \cdot H > 0$ but nD is not linearly equivalent to an effective divisor for any positive integer n.

b) In your example, what is D^2?

c) Can there be an example in part (a) for which $D^2 > 0$?

5. Let X be the blow-up of $\mathbb{P}^2$ at a point P.

a) Describe $\operatorname{Pic} X$ as a group together with the intersection pairing. Determine which divisors on X are ample.

b) Determine which divisors on X are ample.

c) Verify the Hodge Index Theorem for X.

6. Consider the rational ruled surface X, mapping onto the projective x-line $\mathbb{P}^1$, which is constructed as follows. Over the affine x-patch U_0 of $\mathbb{P}^1$, the inverse image is $U_0 \times \mathbb{P}^1$, where the second factor is the projective y-line. Over the affine $\bar{x}$-patch U_1 of $\mathbb{P}^1$ (where $x\bar{x} = 1$ on $U_{01} := U_0 \cap U_1$), the inverse image is the projective $\bar{y}$-line. Over U_{01}, we have the transition function $\bar{y} = \bar{x}^n y$, for some non-negative integer n. Find the numerical invariant e of the rational ruled surface X, and find a locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^1$ such that $X \approx \mathbb{P}(\mathcal{E})$.

7. a) Let H be the line at infinity in $\mathbb{P}^2$, and let P, Q be distinct points on H. Let X be the blow-up of $\mathbb{P}^2$ at P and Q; let E_1, E_2 be the exceptional divisors over P, Q; and let L be the proper transform of H. Describe $\operatorname{Pic} X$, in particular giving the self-intersections of E_1, E_2, L, and giving the pairwise intersections of these three divisors.

b) Let H_1, H_2 be the two lines at infinity in $\mathbb{P}^1 \times \mathbb{P}^1$, given by $x = \infty$ and $y = \infty$ respectively. Let O be the point at which H_1, H_2 intersect. Let X' be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at O; let E be the exceptional divisor over O; and let L_1, L_2 be the proper transforms of H_1, H_2. Describe $\operatorname{Pic} X'$, in particular giving the self-intersections of L_1, L_2, E, and giving the pairwise intersections of these three divisors.

c) Show that $X \approx X'$. Under your isomorphism, which divisors of X' do $E_1, E_2, L \subset X$ correspond to?