1. Let $R$ be a Noetherian domain, and let $\mathfrak{p}$ be a non-zero prime ideal of $R$. Let $R_\mathfrak{p}$ be the local ring of $R$ at $\mathfrak{p}$. Let $\hat{R}_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $R$, viz. $\lim_{\rightarrow} R/\mathfrak{p}^n$.

(a) Show that if $\mathfrak{p}$ is maximal, then there is a natural inclusion $R_\mathfrak{p} \hookrightarrow \hat{R}_\mathfrak{p}$.
(b) Show that this conclusion does not necessarily hold for more general prime ideals $\mathfrak{p}$. [Hint: Try $R = k[x, y]$ and $\mathfrak{p} = (x)$. Is $y$ a unit?]

2. (a) Let $q$ be an odd prime power, and let $K_\infty = \mathbb{F}_q((t^{-1}))$, the infinite completion of $R = \mathbb{F}_q[t]$. Let $\mu = \{\text{roots of 1 in } R\}$. Verify the following:

(i) There is an element $s \in R$ whose norm is $N(s) = (#\mu) + 1$, and having the following property: Every element of $K_\infty$ can uniquely be written in the form $\alpha = \sum_{i=-\infty}^{n} a_is^i$ for some integer $n$, with each $a_i \in \mu \cup \{0\}$ and $a_n \neq 0$. [Hint: There is a very simple choice for $s$.]
(ii) If $\alpha$ is as in (i), then $\alpha$ is a square in $K_\infty$ if $a_n$ is a square in $\mu$ and $n$ is even. [Hint: If $n$ is even, is $s^{-n}\alpha$ a square?]
(iii) Let $S \subset K_\infty$ consist of the elements whose expressions involve only negative powers of $s$. Then $S$ is a fundamental domain for the translation action of $R$ on $K_\infty$.

(b) Now let $K_\infty = \mathbb{R}$, the infinite completion of $R = \mathbb{Z}$. What is the analog of part (a)? [Hint: There are some differences, especially in (ii).] In the analog of (a)(iii), draw $S$.
(c) Redo (b) for $R = \mathbb{Z}[i]$ and for $R = \mathbb{Z}[\zeta_3]$, where $\zeta_3$ is a primitive cube root of unity. [Note: The pictures of $S$ should be a surprise.]

3. (a) Find all $\alpha \in \mathbb{Z}$ such that $\alpha^2 \equiv 2 \pmod{21}$. Then do the same for $\alpha \in \mathbb{Z}[i]$. (Hint: Chinese Remainder Theorem.)

(b) Find all $f(t) \in \mathbb{F}_1[t]$ such that $f(t)^2 \equiv t \pmod{t^2 - 1}$.

4. (a) Find the continued fraction expansion for $\sqrt{7}$.
(b) Find all the units in $\mathbb{Z}(\sqrt{7})$, the ring of integers of $\mathbb{Q}(\sqrt{7})$.
(c) What happens in parts (a) and (b) if $\sqrt{7}$ is replaced by $\sqrt{-7}$?

5. Let $k$ be a field, and let $f(t) = t^4 + t^3 + t^2 + t$ and $g(t) = t^3 + t^2 + 1$ in $k[t]$.

(a) Find the g.c.d. of the polynomials $f(t)$ and $g(t)$ in $k[t]$.
(b) Find polynomials $X(t), Y(t) \in k[t]$ such that $f(t)X(t) + g(t)Y(t) = 1$.
(c) Find the continued fraction expansion for $f(t)/g(t)$ over $k[t]$.

6. Let $k = \mathbb{F}_3$, $R = k[t]$, and $K_\infty = k((t^{-1}))$.

(a) Show that $\sqrt{1 + t^2} \in K_\infty$.
(b) Find the continued fraction expansion for $\sqrt{1 + t^2}$ over $R$.
(c) Find all solutions $X(t), Y(t) \in R$ to the equation $X(t)^2 - (1 + t^2)Y(t)^2 = 1$.
(d) Find all the units in the ring $k[t, \sqrt{1 + t^2}]$, and interpret your answer in terms of functions on the affine curve $y^2 = 1 + x^2$ over $k$.
(e) What happens if $1 + t^2$ is replaced by $1 + t$, or by $1 - t^2$? What goes wrong, and why? What would analogous examples be with $R, K_\infty$ replaced by $\mathbb{Z}, \mathbb{R}$?