1. Let $Y \to \mathbb{P}^1_\mathbb{C}$ be a $G$-Galois branched cover, with branch locus $P_1, \ldots, P_r$, where $P_j$ is at $x = j$. Let $P$ be a base point on the positive imaginary axis. Choose a homotopy basis $\sigma_1, \ldots, \sigma_r$ of counterclockwise loops at $P$, where $\sigma_j$ winds once around $P_j$, and where the loops (and their interiors) are disjoint except at $P$. Let $(g_1, \ldots, g_r)$ be the description of $Y \to \mathbb{P}^1_\mathbb{C}$ with respect to the loops $\sigma_j$. Let $\mathcal{H}$ be the associated Hurwitz space, and let $\xi \in \mathcal{H}$ be the point corresponding to $Y \to \mathbb{P}^1_\mathbb{C}$. Pick a $j$ with $1 \leq j < r$, and let $C_j$ be the circle of radius $\frac{1}{2}$ centered at $x = j + \frac{1}{2}$.

   a) Consider a path $\theta_j$ in $(\mathbb{P}^1)^r$ beginning at $(P_1, \ldots, P_r)$, in which the $j^{\text{th}}$ and $(j+1)^{\text{th}}$ branch points of $Y$ each move counterclockwise along half of $C_j$, and the other branch points remain fixed. (Thus the final point of $\theta_j$ is $(P_1, \ldots, P_j-1, P_j+1, P_j, P_j+2, \ldots, P_r)$.) Lift $\theta_j$ to a path $\Theta_j$ in $\mathcal{H}$ with initial point $\xi$, and let $Y_j \to \mathbb{P}^1_\mathbb{C}$ be the cover corresponding to the final point of $\Theta_j$. Show that the description of $Y_j \to \mathbb{P}^1_\mathbb{C}$, relative to $\sigma_1, \ldots, \sigma_r$ (in that order), is $(g_1, \ldots, g_{j-1}, g_{j+1}, g_j^{-1} g_j g_{j+1}, g_{j+2}, \ldots, g_r)$. [Hint: Verify all the entries of the description other than $g_{j+1}^{-1} g_j g_{j+1}$, and then show that that one is forced.]

   b) Now consider the loop $\theta_j'$ in $(\mathbb{P}^1)^r$ at $(P_1, \ldots, P_r)$, obtained by “doing $\theta_j$ twice.” Lift $\theta_j'$ to a path $\Theta_j'$ in $\mathcal{H}$ with initial point $\xi$, and let $Y_j' \to \mathbb{P}^1_\mathbb{C}$ be the cover corresponding to the final point of $\Theta_j'$. Show that the description of $Y_j' \to \mathbb{P}^1_\mathbb{C}$, relative to $\sigma_1, \ldots, \sigma_r$, is $(g_1', \ldots, g_r')$, where $g_l' = g_l$ for $l \neq j, j+1$, and $g_j' = (g_j g_{j+1})^{-1} g_l g_j g_{j+1}$ for $l = j, j+1$. [Hint: Iterate part (a).]

   c) Show that if the cover $Y \to \mathbb{P}^1_\mathbb{C}$ is deformed by allowing the $(j+1)^{\text{th}}$ point to wind once around $P_j$ counterclockwise, then the resulting cover has description $(g_1', \ldots, g_r')$ as in (b). Show that the same happens in instead we allow the $j^{\text{th}}$ point to wind once around $P_{j+1}$ clockwise. What happens if the $j^{\text{th}}$ point winds once around $P_{j+1}$ counterclockwise? [Hint: Use part (b).]

2. In problem 1, let $G = S_3$, $r = 4$, and $(g_1, \ldots, g_4) = ((12), (12), (13), (13))$. Thus $Y \to \mathbb{P}^1_\mathbb{C}$ is a slit cover. Let $P_0$ be the point $(x = 0)$. Consider paths $\theta$ in $(\mathbb{P}^1)^4$ with initial point $(P_1, P_2, P_3, P_4)$ and final point $(P_1, P_0, P_3, P_0)$, such that $\theta(t) \in (\mathbb{P}^1)^4 - \Delta$ for $0 \leq t < 1$, and such that the first and third entries of $\theta(t)$ remain constant as $t$ varies.

   a) Show that each such path $\theta$ lifts to a unique path in the compactified Hurwitz space $\overline{\mathcal{H}}$ beginning at the point $\xi \in \mathcal{H}$ that corresponds to $Y \to \mathbb{P}^1$. Explain why the final point of the lifted path determines an unramified cover $Y_\theta \to \mathbb{P}^1 - \{P_0, P_1, P_3\}$.

   b) Find a choice of $\theta$ such that $Y_\theta$ is connected, and find another choice of $\theta$ such that $Y_\theta$ is not connected. [Hint: Choose a homotopy basis for $\mathbb{P}^1 - \{P_0, P_1, P_3\}$, and use problem 1 to determine the description of a $Y_\theta$. From the description, how can you tell if a cover is connected?]

3. Let $Y \to \mathbb{P}^1$ be as in problem 2, and let $\mathcal{P}$ be the space that parametrizes the covers obtained from $Y \to \mathbb{P}^1$ by allowing the third branch point to wander in $\mathbb{P}^1 - \{P_1, P_2, P_4\}$, while holding the other three branch points fixed. Let $\xi \in \mathcal{P}$ be the point corresponding to the cover $Y \to \mathbb{P}^1$.

   a) Describe an unramified covering map $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$, and find the image
of $\xi$.

b) Show that $\pi$ is not an isomorphism. [Hint: Use problem 1 to show that $\deg(\pi) > 1$.]

c) Let $\pi : \mathcal{P} \to \mathbb{P}^1$ be the branched cover obtained by compactifying $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$. Show that there are points $\eta, \eta' \in \mathcal{P}$ lying over $P_4 \in \mathbb{P}^1$ such that $\pi$ is unramified at the point $\eta$ but is ramified at $\eta'$. [Hint: Show that a loop $\tau$ at $P_3$ around $P_4$ takes $Y$ to itself, but that $\tau$ takes some other cover in the family (also with branch locus $P_1, P_2, P_3, P_4$) to a cover other than itself.]

d) Deduce that $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$ is not Galois. [Hint: Use part (c).]

e) Conclude that the Hurwitz space associated to an $S_3$-Galois mock cover with generators $\{(12), (13)\}$ is not Galois over $(\mathbb{P}^1)^4 - \Delta$. [Hint: Relate the Hurwitz space $H \to (\mathbb{P}^1)^4 - \Delta$ to the cover $\mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$.]

4. Let $k$ be a field of characteristic $\neq 2$. Let $X = \mathbb{P}^1_{k[[t]]}$ (the projective $x$-line over $k[[t]]$), let $X_1 = \text{Spec } k[x][[t]]$, let $X_2 = \text{Spec } k[x][[t]]$ (where $x \bar{x} = 1$), and let $X_0 = \text{Spec } k[x, x^{-1}][[t]]$.

a) Prove that there is a 2-cyclic Galois branched cover of $X$ whose pullback to $X_1$ is given by $y^2 = x(x - t)$; whose pullback to $X_2$ is given by $z^2 = \bar{x}(\bar{x} - t)$; and whose pullback to $X_0$ is trivial (i.e. a disjoint union of two copies of $X_0$).

b) Find such a cover explicitly.

5. For any ring $A$ with an absolute value $| \cdot |$, let $A \{x\} \subset A[[x]]$ denote the ring of power series that converge on $|x| \leq 1$, and let $A \langle x \rangle \subset A[[x]]$ denote the ring of power series that converge on $|x| < 1$. Let $k$ be a field, let $R = k[[t]]$, and let $K = k((t))$.

a) Show that $R \{x\} = k[x][[t]]$ and $K \{x\} = k[x][[t]][t^{-1}]$.

b) Show that $R \langle x \rangle = k[[x, t]]$ and $K \langle x \rangle = k[[x, t]][t^{-1}]$. 